Theoretical and experimental demonstration of the transient responses of a circuit

Practice proposed for Circuit Analysis, course 2016-17.
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Problem

The scheme of the circuit shown in figure 1 corresponds to the real circuit shown in figure 2.

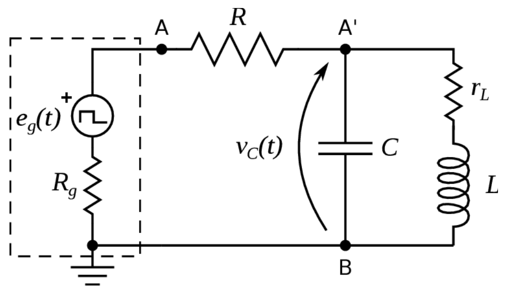


Figure 1. Scheme of the circuit.

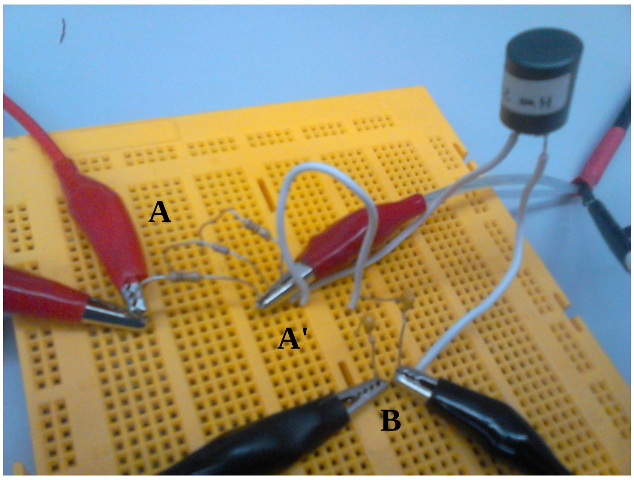


Figure 2. Experimental set-up of the circuit.

The circuit is connected to an external voltage source that applies the following “rec” function

$$e_g(t) = E_0\left(u(t) - u(t - \frac{T}{2})\right), \tag{1}$$

where T is the period of the signal generator which has a frequency of f, so that T=1/f, and the step height is E_0. The function generator has an internal resistance R_g=50Ω. The coil of inductance L=22mH, has also an internal resistance r_L=25Ω. For a given value of f and E_0, the values of the resistance, R, and the capacitance, C, are changed to see how they will affect the transient behavior of the voltage measured at the capacitor, $v_C(t)$.

The value of the capacitance, C, can be either 470nF or 10nF.

The value of the resistance, R, can be 0, 100Ω, 1000Ω or 10000Ω.

For the theoretical solution it is to consider that time the voltage E_0 is applied is much longer than the time the transient behavior happens ($T/2 >> \tau$).

Theoretical solution

If $T/2 >> \tau$ then the initial conditions are close to zero: $v_C(0)$=0, $i_L(0)$=0.

The circuit is then solved for $t \geqslant 0$ and the resulting Laplace transform of the voltage at the capacitor is:

$$V_C(s) = K\frac{\alpha + s}{s(s^2 + a_1 s + a_0)}\left(1 - \mathrm{e}^{-\frac{T}{2}s}\right), \tag{2}$$

where

$$K = \frac{E_0}{C(R+R_g)},$$

$$\alpha = \frac{r_L}{L},$$

$$a_1 = \left(\frac{r_L}{L} + \frac{1}{C(R+R_g)}\right),$$

$$a_0 = \left(1 + \frac{r_L}{R+R_g}\right)\frac{1}{LC}.$$

To obtain the inverse Laplace transform ($\mathfrak{L}^{-1}$) of $V_C(s)$, the following property can be used:

If $V_C(s) = G(s)(1 - \mathrm{e}^{-\frac{T}{2}s})$, then

$$v_C(t) = g(t)u(t) - g(t - \frac{T}{2})u(t - \frac{T}{2}), \tag{3}$$

being $g(t) = \mathfrak{L}^{-1}[G(s)]$. Thus, it only remains to obtain the Laplace transform of

$$G(s) = K\frac{\alpha + s}{s(s^2 + a_1 s + a_0)}. \tag{4}$$

From the characteristic polynomial that appears in eq.(4), the natural frequency, ω_n, and the damping ration, ξ, can be obtained:

$$\omega_n = \sqrt{a_0}, \tag{5}$$

$$\xi = \frac{a_1}{2\sqrt{a_0}}. \tag{6}$$

Depending on the values of damping ration, ξ, or else, depending on the roots $\{s_1, s_2\}$ of the characteristic polynomial $s^2 + a_1 s + a_0 = 0$, the following different type of solutions are obtained:

1. **Under-damped**, when $0 < \xi < 1$, and the roots are complex conjugated $s_1 = -a + jb$, $s_2 = -a + jb$, being $a = \xi\omega_n$, $b = \omega_n\sqrt{1-\xi^2}$. The following inverse Laplace transform (Nr. 28 of the table) is obtained:

$$g(t) = A + B\mathrm{e}^{-at}\sin(bt + \phi), \tag{7}$$

where

$A = \dfrac{K\alpha}{a^2 + b^2} = g(t \to \infty)$, which is the voltage at r_L.

$$B = \frac{K}{b}\sqrt{\frac{(\alpha - a)^2 + b^2}{a^2 + b^2}},$$

$$\phi = \arctan\frac{b}{\alpha - a} - \arctan\frac{b}{-a}.$$

How fast the under-dampet transient behavior decreases is given by the inverse of the time constant or settling time, τ, obtained from the real part of the roots of the characteristic polynomial. For the under-daped systems:

$$\tau = \frac{1}{a} = \frac{1}{\xi\omega_n}, \tag{8}$$

This settling time is always larger the one obtained for the over-damped and critically-damped solutions. And the oscillation period is given by the imaginary part of the roots:

$$T_u = \frac{2\pi}{b} = \frac{2\pi}{\omega_n\sqrt{\xi^2 - 1}}, \tag{9}$$

2. **Over-damped**, when $\xi > 1$, and the roots $s_1 = -a, s_2 = -b$ are real, negative and different, being $a = \xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$, $b = \xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$. The following inverse Laplace transform (Nr. 11 of the table) is obtained:

$$g(t) = A + B\mathrm{e}^{-at} + D\mathrm{e}^{-bt}, \tag{10}$$

where

$$A = \frac{K\alpha}{ab},$$

$$B = -\frac{Kb(\alpha - a)}{a(b - a)},$$

$$D = \frac{K(\alpha - b)}{b(b - a)}.$$

It is easy to check that $g(t = 0) = A + B + D = v_C(0) = 0$.

3. **Critically damped**, when $\xi = 1$, and the roots are real and equal, $s_1 = s_2 = -a = -\xi\omega_n$. The obtained inverse Laplace transform (Nr. 34) is:

$$g(t) = A + (B + Dt)\,\mathrm{e}^{-at}. \tag{11}$$

where

$$A = \frac{K\alpha}{a^2},$$

$$B = -A \Rightarrow g(0) = 0,$$

$$D = \frac{K(a - \alpha)}{a}.$$

The value of *R* for which the solution is critically damped (when $a_1 = 2\sqrt{a_0}$) is

$$R_c = 2\left[2\alpha C + \sqrt{(2\alpha C)^2 - 4\frac{C}{L}(\alpha C - 4)}\right]^{-1} - R_g. \tag{12}$$

Experimental set-up

The function generator is shown in figure 3, and is connected to the circuit shown in figure 2 at the points A and B (B is GND), and the channel 1 of the oscilloscope (shown in figure 4) is connected to the points A' and B. To avoid flickering and replications of the visualized signal, it is necessary to introduce a trigger signal into the oscilloscope provided by the same signal of the function generator. The input signal of the function generator can also be visualized by connecting the channel 2 of the oscilloscope to the points A and B.

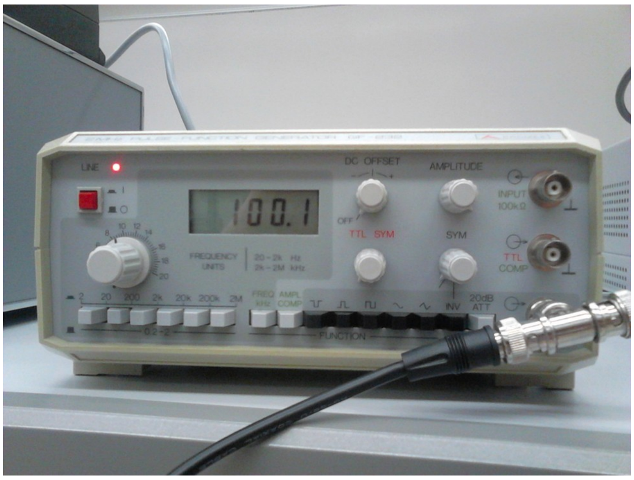


Figure 3. Function generator.

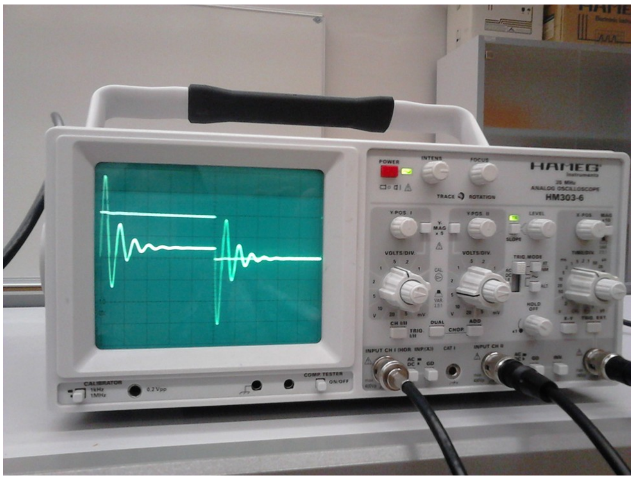


Figure 4. Oscilloscope.

Comparison between theoretical and experimental results

The theoretical representations of eq.(3) (with their corresponding type of solutions for $g(t)$ given in eq.(7), (10) or (11)) are obtained with MATLAB so that the time evolution of $v_C(t)$ is represented as if it where displayed by the screen of the oscilloscope, according to the VOLTS/DIV. and TIME/DIV. adjustments, but the time interval is restricted between 0 and $T=1/f$.

Generate a “rec” signal (eq. (1)) using the function generator with, for example, E_0 =10V and f=100 Hz. (In the real experiment adjust the voltage with channel 1 of the oscilloscope before connecting the function generator to the circuit).

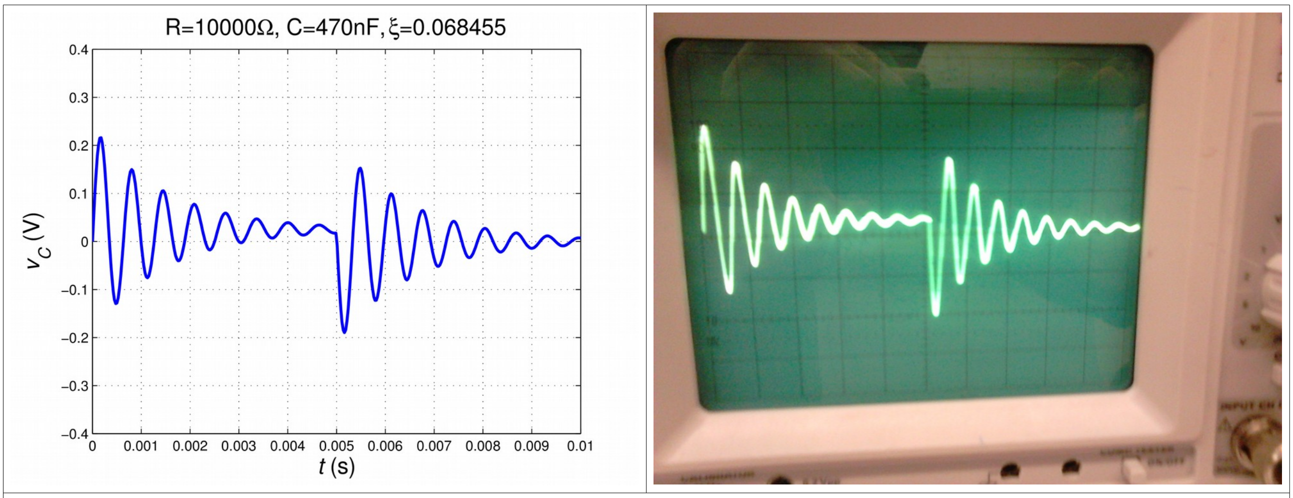


Figura 5(a). VOLT/DIV=0.1V, TIME/DIV=1ms, f=100Hz.

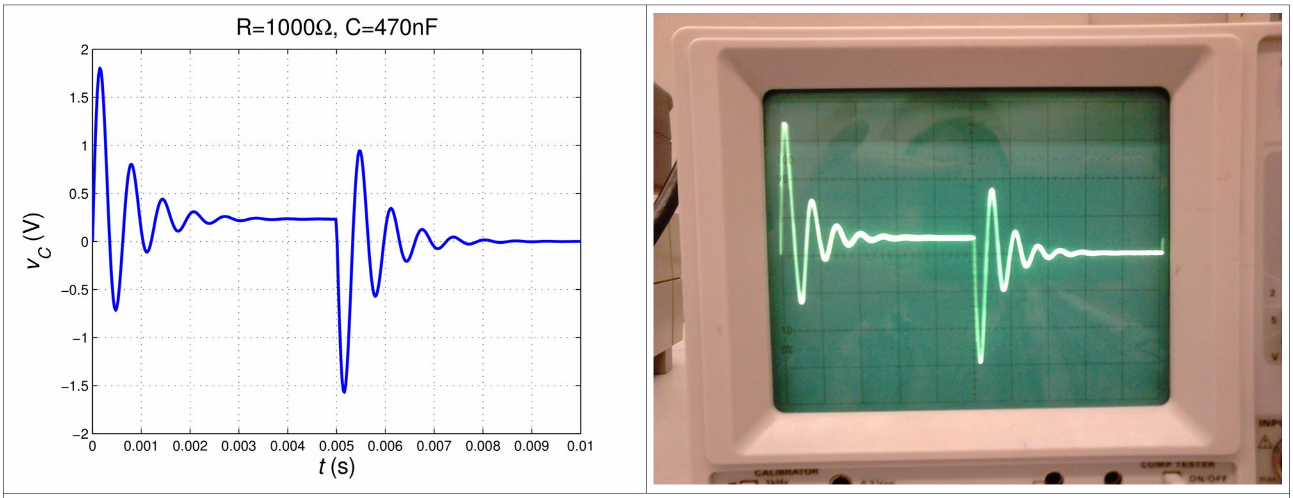


Figura 5(b). VOLT/DIV=0.5V, TIME/DIV=1ms, f=100Hz.

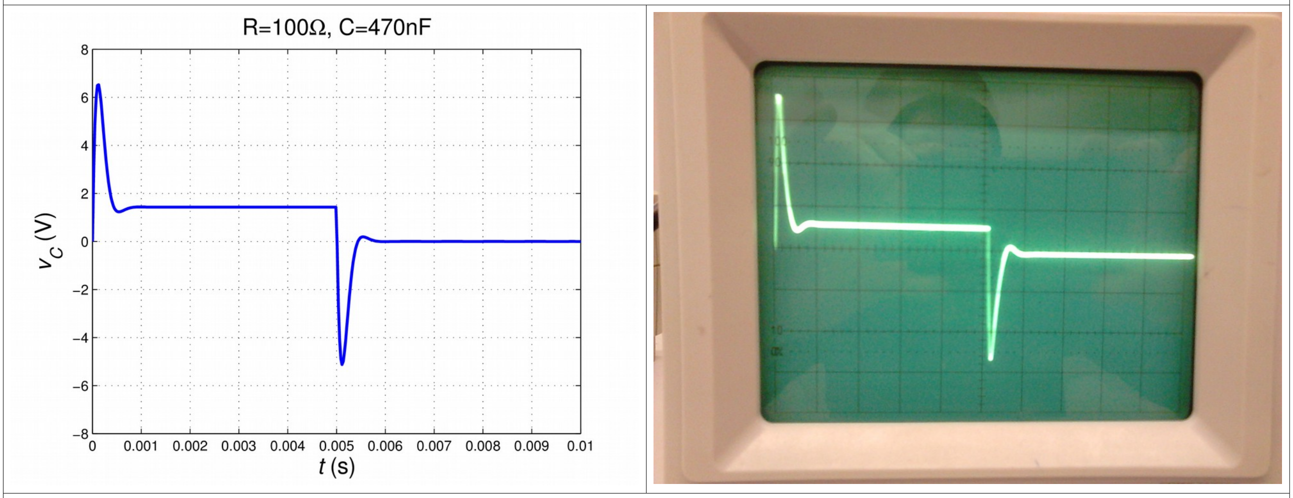


Figura 5(c). VOLT/DIV=2V, TIME/DIV=1ms, f=100Hz.

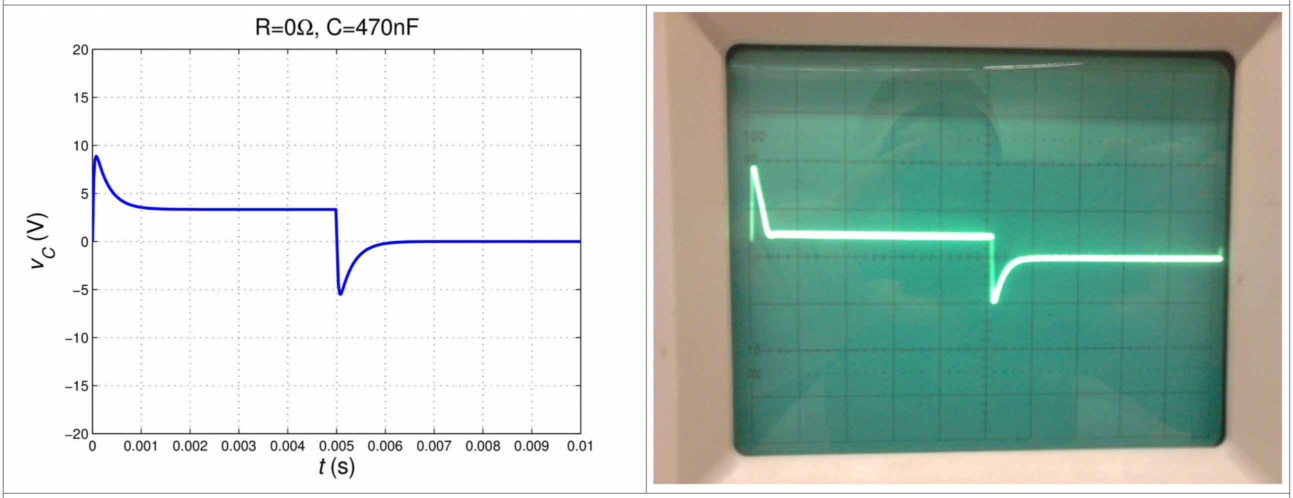
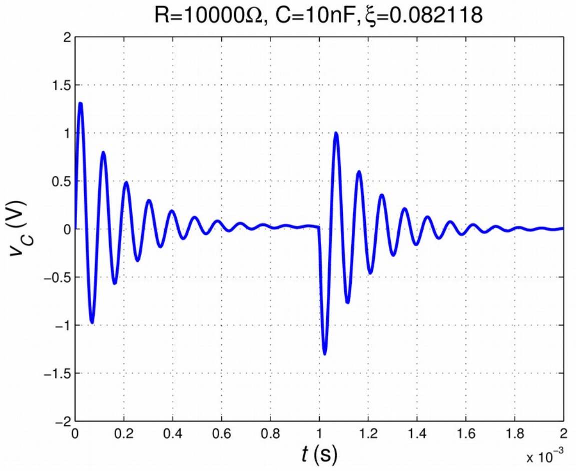


Figura 5(d). VOLT/DIV=5V, TIME/DIV=1ms, f=100Hz.



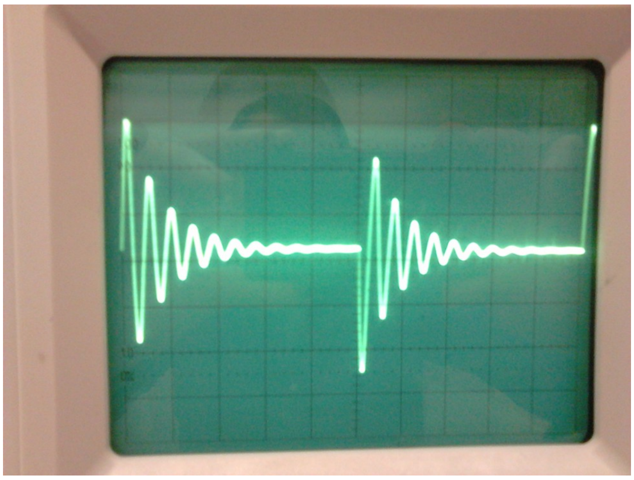
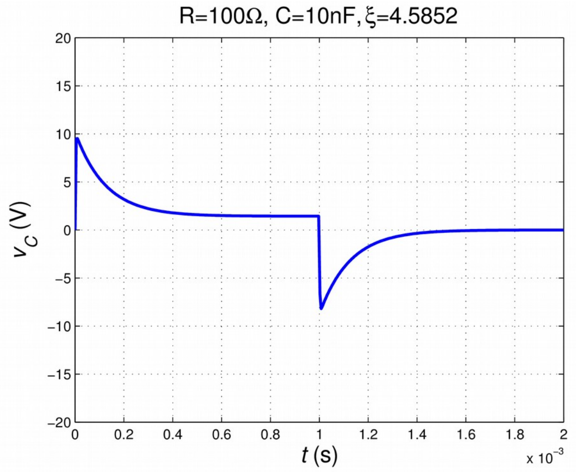


Figura 5(e). VOLT/DIV=0.5V, TIME/DIV=0.2ms, f=500Hz.



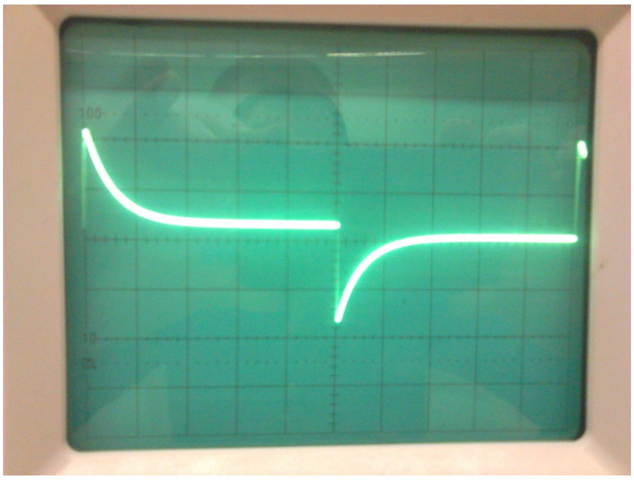


Figura 5(f). VOLT/DIV=5V, TIME/DIV=0.2ms, f=500Hz.

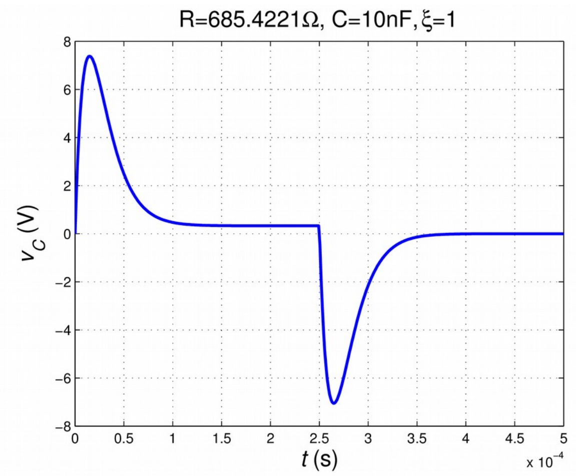


Figura 5(g). VOLT/DIV=2V, TIME/DIV=50µs, f=2000Hz.

Matlab program

The following program for MATLAB® can be used for graphically representing the theoretical obtained solutions. (Copy and paste in your .m file).

```
%% Transient_response_Experiment
% Visualisation of the transient responses of a rectangular pulse.
% The theoretical results of the corresponding experiment are computed here
% to demostrate that the theory can be used to explain and predict the real
% fenomena.
%% Experimental Set-up:
% Function generator {eg(t)+Rg} connected to {R + C//{rL+L}}
% Oscilloscope (e.g. channel 1) connected to C: vC(t) is displaied
% (Ground connection have to be connected to the same ground of the source)
% The oscilloscope has to be trigered with the output signal of the
% fucntion generator.
% Channel 2 of the oscilloscope can be connected to the source output to
% also visualice the applied external "Rec" signal

%% DATA
%Circuits components
L=22*10^(-3); %H
rL=25; %Ohm, internal resistance of L: {rL+L}
%C=470*10^(-9);
C=10*10^(-9);
R=10000; %Ohm
%R=1000;
%R=100;
%R=685.4221
%R=0
%Funciton generator: "Rec" pulse: eg(t)=E0*(u(t)-u(t-T/2)), T=1/f
E0=10; %V
f=500; %Hz
Rg=50; %Ohm, internal resistance of the functio generator {E+Rg}
%Osciloscope set-up:
Tdiv=0.2*10^(-3); %s
Vdiv=0.5; %V

%% SOLUTIO
%Voltage at C: vC(t)

%Time interval for vC(t) is one period of the function generator
T=1/f;
t = [0:T/399:T];

%The Laplace transform of vC(t):
%VC(s)=K*(s+alpha)*(1-exp(-T*s/2))/[s*(s^2+a1*s*a0)],
%being:
```

```
K = E0./(C*(R+Rg));
alpha = rL./L;
a1 = rL/L + 1./(C.*(R+Rg));
a0 = (1+rL/(R+Rg))./(L*C);

wn = sqrt(a0); %Natural frequency
D = a1./(2*wn) %Damping ratio
tau = 1/(D*wn) %Settling time

%Value of R for Critically damped solution (when a1=2*sqrt(a0))
a = -2*rL*C/L; b = (rL*C/L)^2-4*C/L;
Rc(1) = 2/(-a + sqrt(a^2-4*b))-Rg; Rc(2) = 2/(-a - sqrt(a^2-4*b))-Rg;
Rc = Rc(Rc>0) %it is the positive solution
if isempty(Rc)==1
   display('there is no R to get critically damped solution')
end
clear a b

%The inverse Laplace transform L⁻¹ {}
%G(s)=K*(s+alpha)/[s*(s^2+a1*s*a0)] -> g(t)=L⁻¹{G(s)}
%VC(s)=G(s)*(1-exp(-T*s/2)) -> vC(t)=g(t)u(t)-g(t-T/2)u(t-T/2);
u=ones(size(t)); %u=u(t)
uT=u; uT(1:round(length(t)/2))=0; %uT=u(t-T)
if D<1; %s1 and s2 are complex conjugated (Underdamped)
   a = D.*wn;
   b = wn*sqrt(1-D^2); %Damped natural frequency
   %s1=-a+i*b; s2=-a-i*b;
   phi = atan2(b,alpha-a)-atan2(b,-a);
   A = K*alpha/(a^2+b^2);
   B = (K/b).*sqrt(((alpha-a)^2+b^2)/(a^2+b^2));
   vC1 = (A + B.*exp(-a.*t).*sin(b.*t+phi)).*u; %table Nr. 28
   vC2 = - (A + B.*exp(-a.*(t-T/2)).*sin(b.*(t-T/2)+phi)).*uT;
   vC2(isnan(vC2)==1)=0;
   vC=vC1+vC2;
elseif D>1 %s1 and s2 are real, negative and different (Overdamped)
   a = D*wn-wn.*sqrt(D^2-1); %a=-s1
   b = D*wn+wn.*sqrt(D^2-1); %b=-s2
   A = b.*(alpha-a)./(b-a);
   B = a.*(alpha-b)./(b-a);
   vC1 = (K/(a*b)).*(alpha - A.*exp(-a.*t) + B.*exp(-b.*t)).*u; %table Nr. 11
   vC2 = -(K/(a*b)).*(alpha - A.*exp(-a.*(t-T/2)) + ...
      B.*exp(-b.*(t-T/2))).*uT;
   vC2(isnan(vC2)==1)=0;
   vC=vC1+vC2;
elseif D==1 %s1=s2 are real, negative and equal (Critically damped)
   a = D.*wn; %s1=-a
   vC1 = (K/a^2).*(alpha - alpha.*exp(-a.*t) + ...
      a*(a-alpha).*t.*exp(-a.*t)).*u; %table Nr. 34
   vC2 = (K/a^2).*(alpha - alpha.*exp(-a.*(t-T/2)) + ...
      a*(a-alpha).*(t-T/2).*exp(-a.*(t-T/2))).*uT;
```

```
    vC2(isnan(vC2)==1)=0;
    vC=vC1+vC2;
end
%(The undamped solution can not be obtained with the experiment)

%% Representation (con ajustes del osciloscopio)
    figure
    plot(t,vC,'LineWidth',2)
    axis([0 10*Tdiv -4*Vdiv 4*Vdiv])
    grid on
    xlabel('{\itt} (s)','FontSize',16)
    ylabel('{\itv_C} (V)','FontSize',16)
    title(['R=',num2str(R),'\Omega, C=',num2str(C*10^9),...
        'nF, \xi=',num2str(D),', \tau=',num2str(tau),'s'],'FontSize',16)
```

Questions

1. Deduce equation (2).

Consider the following cases:

a) R=10kΩ, C=470nF.

b) R=1kΩ, C=470nF.

c) R=100Ω, C=470nF.

d) R=0, C=479nF

e) R=10kΩ, C=10nF.

f) R=100Ω, C=10nF.

g) R=685.42Ω, C=10nF.

2. Obtain the temporal expression of $v_C(t)$ for the cases e), f), and g).

3. Perform the experiments of the measurements of $v_C(t)$ for the cases a) to f), take a picture from the screen of the oscilloscope. Compare them with the theoretically obtained results and answer the following questions:

   3.1 For the same value of C, does the damping ratio (ξ) increase or decrease for smaller R?. How can this be explained?.

   3.2 How affects the value of C to the oscillation for the under-damped cases.

   3.3 Way decreases the transient response of $v_C(t)$ for $0<t<T/2$ not to zero?. (or, the voltage at C tends to the voltage at ¿…?)

7. Considering C=470nF, what is the value of R that produces the critically damped behavior.

8. What happens when the frequency of the function generator increases?. (or, what happens when $T/2$ becomes closer to τ?)