U1. Transient circuits response

Circuit Analysis, Grado en Ingeniería de Comunicaciones
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- Cause of the transient responses
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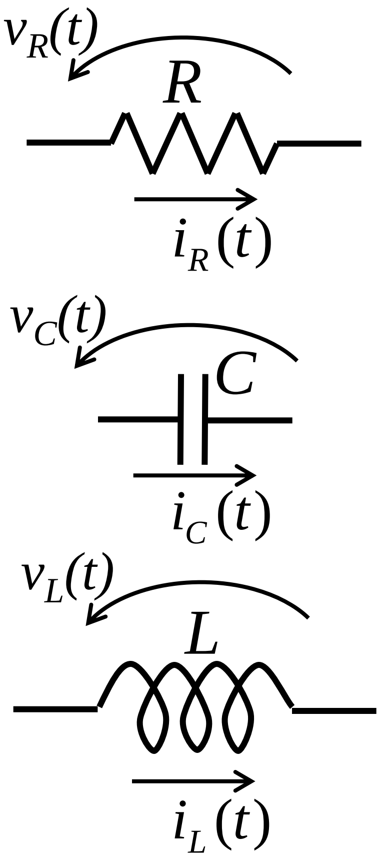
$$\left(v(t) = \frac{dW(t)}{dq}, i(t) = \frac{dq(t)}{dt} \right)$$

Recall

- Relation between *i*(*t*) and *v*(*t*) for the passive elements *R*,*L*,*C*
  - For *R:* $v(t) = Ri(t)$
  - For *C:* $q(t) = Cv_C(t) \Rightarrow i_C(t) = C\dfrac{dv_C(t)}{dt}$
  - For *L:* $v_L(t) = L\dfrac{di_L(t)}{dt}$
- Energy in these elements:
  - Dissipated in *R*: $W_R(t) = R\int i_R^2(t)dt$
  - Stored in *C* and *L*:

$$W_C(t) = \frac{1}{2}C\left(v_C(t)\right)^2 \qquad W_L(t) = \frac{1}{2}L\left(i_L(t)\right)^2$$

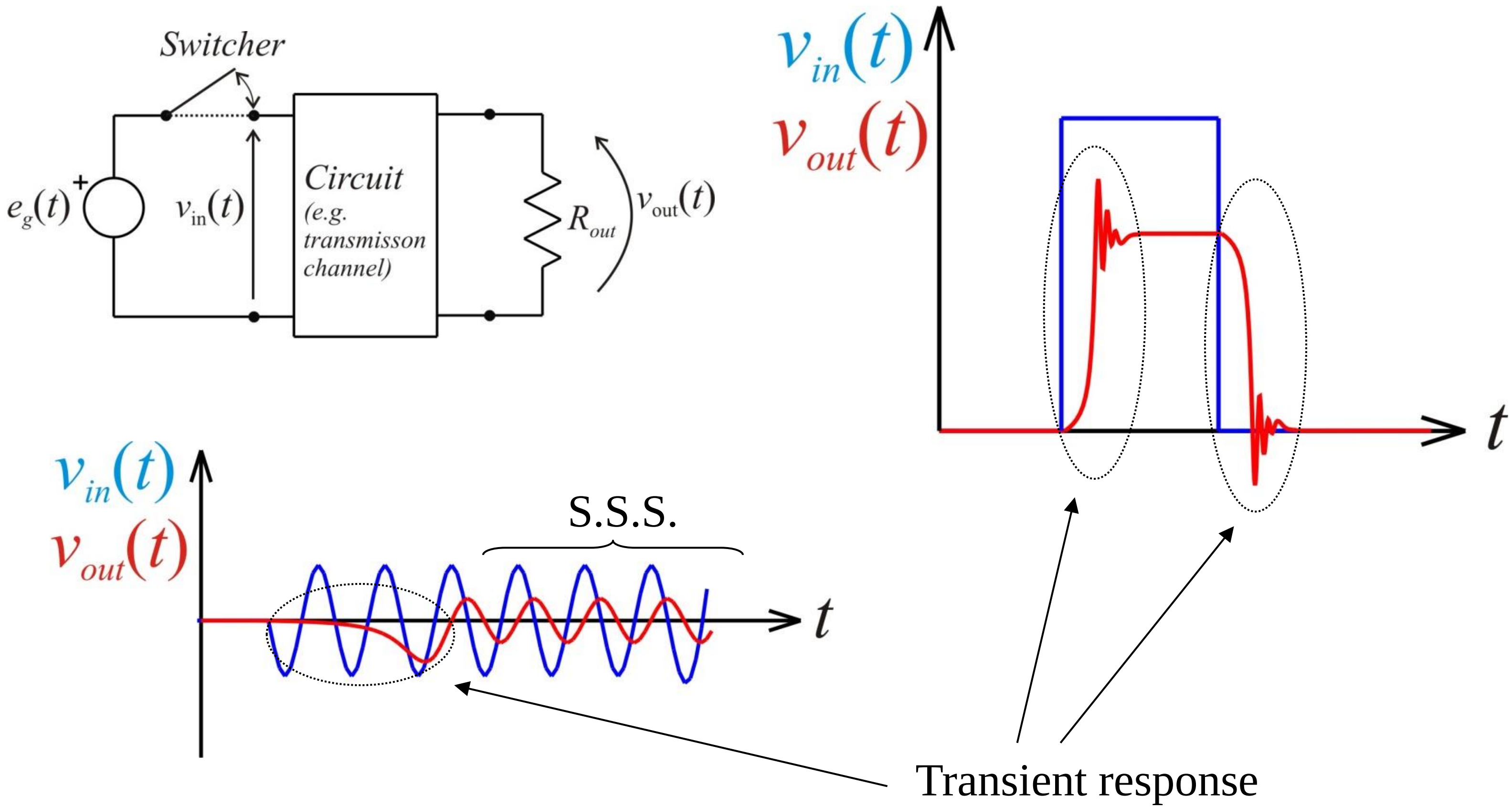


Goals

- We want to solve circuits for whatever applied source (not only DC and Sinusoidal Steady State)
  - Direct resolution in the time domain
  - Resolution using Laplace transforms
- In particular we want to understand what happens when an abrupt change takes place in the circuit, which will produce the transient response.

Motivation

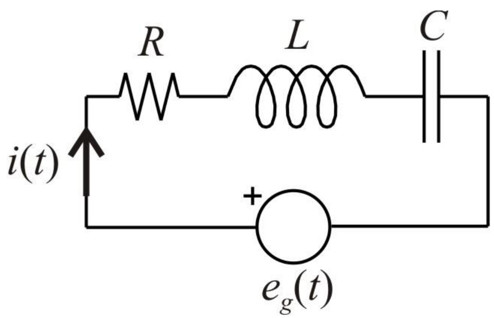
- Signal transmission



Examples of 2nd order circuits

- RLC-serial

(http://en.wikipedia.org/wiki/RLC_circuit)

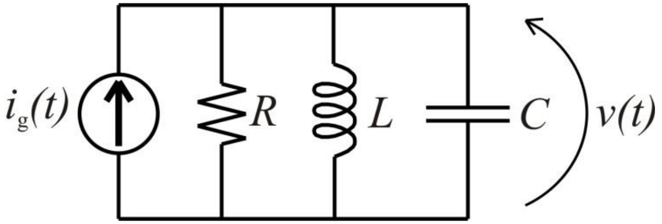


$$v_R(t)+v_L(t)+v_C(t)=e_g(t) \quad \text{(Energy conservation)}$$

$$\Rightarrow Ri(t)+L\frac{di(t)}{dt}+\frac{1}{C}q(t)=e_g(t)$$

$$\Rightarrow \frac{d^2i(t)}{dt^2}+\frac{R}{L}\frac{di(t)}{dt}+\frac{1}{LC}i(t)=\frac{1}{L}\frac{de_g(t)}{dt}$$

- RLC-parallel



$$\frac{d}{dt}\left\{i_R(t)+i_L(t)+i_C(t)=i_g(t)\right\} \quad \text{(Charge conservation)}$$

$$\Rightarrow \frac{1}{R}\frac{dv(t)}{dt}+\frac{1}{L}v(t)+C\frac{d^2v(t)}{dt^2}=\frac{di_g(t)}{dt}$$

$$\Rightarrow \frac{d^2v(t)}{dt^2}+\frac{1}{RC}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=\frac{1}{C}\frac{di_g(t)}{dt}$$

Transient response

- Response of a circuit (voltage or current) when an abrupt change happens (e.g. switching)
- Time evolution until achieving a new equilibrium
- The transition function follows exponential variations (decreasing or increasing, fluctuating or no fluctuating)
- They are solutions of linear differential equations

General solution

Linear differential equation of order *n*:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_0 y(t) = g(t),$$

$$y(t) = y_h(t) + y_p(t),$$

$\boldsymbol{y_h(t)}$ is the solution for $g(t)$=0: the Complementary, natural or homogeneous solution. Gives the *transient behavior* of the circuit due to the passive elements. It is dependent of the initial conditions.

$\boldsymbol{y_p(t)}$ is a particular solution for the given source or forcing function $g(t)$. The particular solution looks like the forcing function, e.g.:

- If $g(t)$ is constant, then $y_p(t)$ is constant
- If $g(t)$ is sinusoidal, then $y_p(t)$ is sinusoidal (i.e. the S.S.S.)

The homogeneous solution decreases exponentially so that

$$y(t\to\infty) \to y_p(t)$$

The homogeneous solution

$y_{\mathrm{h}}(t)$ is the solution for $g(t)=0$ (external energy supply =0)

The solution has the form: $y_{\mathrm{h}}(t)=A\cdot\exp(st)$ ("Ansatz")

$$\text{since: } \frac{d^k}{dt^k}\left(A\,\mathrm{e}^{st}\right)=As^k\,\mathrm{e}^{st}$$

$$\Rightarrow s^n + a_{n-1}s^{n-1} + \ldots + a_0 = 0,$$ **characteristic polynomial equation**

$$\Rightarrow s = \{s_1, s_2, \ldots, s_n\}$$ ("Eigenwerte")

$$\Rightarrow y_h(t) = A_1\,\mathrm{e}^{s_1 t} + A_2\,\mathrm{e}^{s_2 t} + \ldots + A_n\,\mathrm{e}^{s_n t} = \sum_{k=1}^{n} A_k\,\mathrm{e}^{s_k t}$$

A_1, A_2, … are obtained with the **initial** (and/or boundary) **conditions**

1st and 2nd order linear differential equations

1st order

$$\frac{dy(t)}{dt} + a_0 y(t) = g(t),$$
$$a_0 = \frac{1}{\tau}$$

For circuits containing one energy storage element (*C* or *L*)

τ is a time constant (how fast $y_h(t)$ decreases)

2nd order

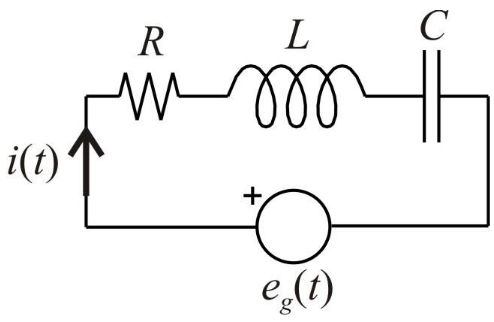
$$\frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = g(t),$$
$$a_1 = 2\xi\omega_n$$
$$a_0 = \omega_n^2$$

For circuits containing two independent energy storage element

ξ is called the **damping ratio** (accounts for the energy loss)
ω_n is called the **natural frequency** (maximum energy storage)
(http://en.wikipedia.org/wiki/Damping)

Example of 2º orden circuits

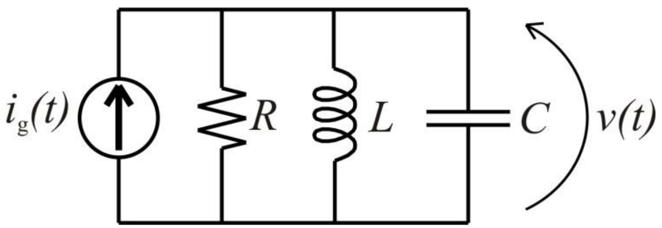
- RLC-serial
  (http://en.wikipedia.org/wiki/RLC_circuit)



$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC} i(t) = \frac{1}{L}\frac{de_g(t)}{dt}$$

$$\Rightarrow \omega_n = \frac{1}{\sqrt{LC}}, \xi = \frac{R}{2\omega_n L}$$

- RLC-parallel

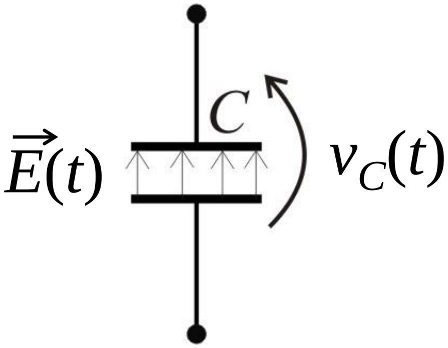


$$\frac{d^2 v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC} v(t) = \frac{1}{C}\frac{di_g(t)}{dt}$$

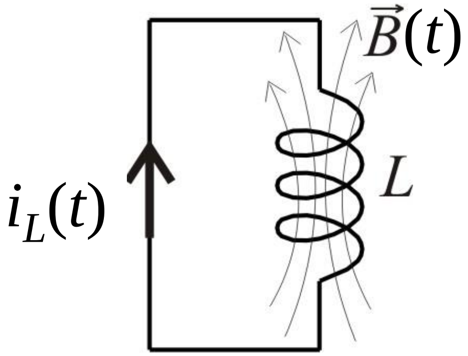
$$\Rightarrow \omega_n = \frac{1}{\sqrt{LC}}, \xi = \frac{1}{2R\omega_n C}$$

Initial conditions

- For each energy storage element we need an initial condition (at $t=t_0$):
  - For C: $v_C(t=t_0) = V_0$
  - For L: $i_L(t=t_0) = I_0$



$$W_L(t) = \frac{1}{2} C\left(v_C(t)\right)^2$$



$$W_L(t) = \frac{1}{2} L\left(i_L(t)\right)^2$$

Condiciones iniciales

- When an abrupt change happens at $t=t_0$ in a circuit, there is always continuity in the variation of the energies in C and $L \Rightarrow$
  - There is continuity in the voltage at C:

$$v_C(t_0^-) = v_C(t_0^+)$$

(not so for the current $i_C(t)$)

Justo antes de t_0 — Justo después de t_0

  - There is continuity in the current trough L:

$$i_L(t_0^-) = i_L(t_0^+)$$

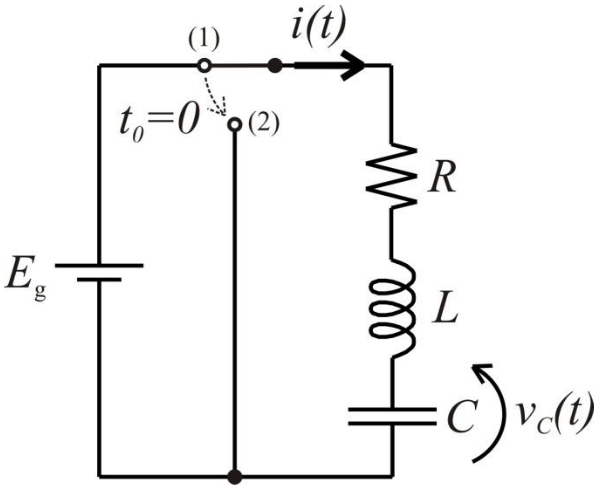
(not so for the voltage $v_L(t)$)

Example of transient response

Discharge of the capacitor

Initial conditions at $t=0$:

$$\begin{cases} i(0)=0 \\ v_C(0)=E_g \end{cases}$$



For $t \geq 0$

$$Ri(t)+\frac{1}{C}q(t)+L\frac{di(t)}{dt}=0,$$

$$\Rightarrow \frac{d^2 i(t)}{dt^2}+\frac{R}{L}\frac{di(t)}{dt}+\frac{1}{LC}i(t)=0,$$

Homogeneus (source free) equaion with solutions of the form

$$i(t)=Ae^{st},$$

Characteristic equation:

$$\Rightarrow s^2+\frac{R}{L}s+\frac{1}{LC}=0$$

Once *i(t)* is known we can obtain $v_C(t)$:

$$v_C(t)=-v_R(t)-v_L(t)=-Ri(t)-L\frac{di(t)}{dt}$$

Type of solutions of the homogeneous equation

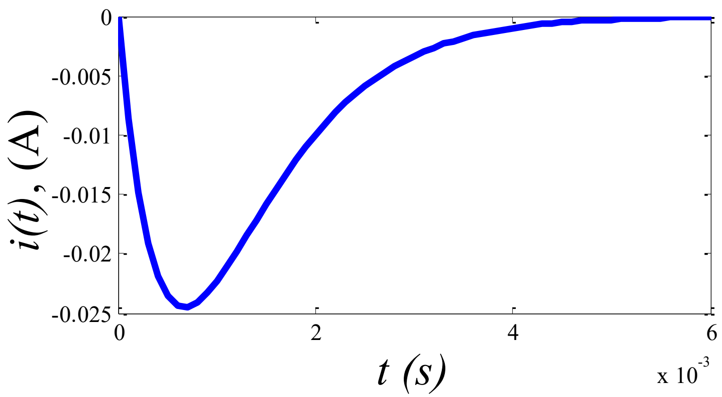
$s^2 + a_1 s + a_0 = 0$ being $a_1 = \frac{R}{L} = 2\xi\omega_n, \quad a_0 = \frac{1}{LC} = \omega_n^2$

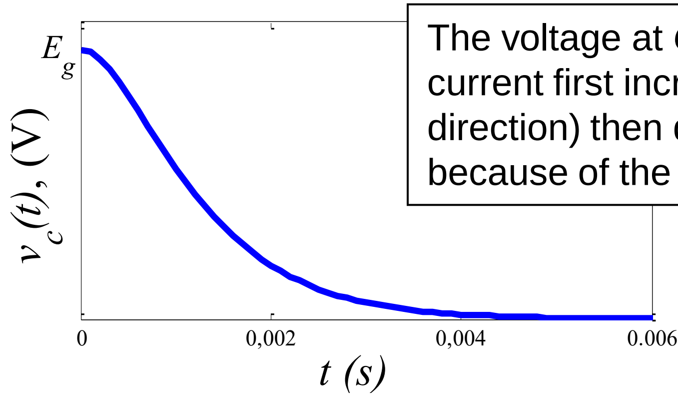
$$s_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}, \text{(units: s}^{-1}\text{)}$$

If $a_1^2 > 4a_0, (\xi > 1)$

$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$: **Overdamped**

$$\left.\begin{aligned} i(0) &= A_1 + A_2 = 0 \Rightarrow i(t) = A_1\left(e^{s_1 t} - e^{s_2 t}\right) \\ v_C(0) &= -Ri(0) - L\left.\frac{di(t)}{dt}\right|_{t=0} = E_g \Rightarrow A_1 = \frac{-E_g}{L(s_1 - s_2)} \end{aligned}\right\} \Rightarrow i(t) = \frac{-E_g\left(e^{s_1 t} - e^{s_2 t}\right)}{L(s_1 - s_2)}.$$





The voltage at *C* decreases. The current first increases (in opposite direction) then decreases to zero because of the energy dissipated in *R*.

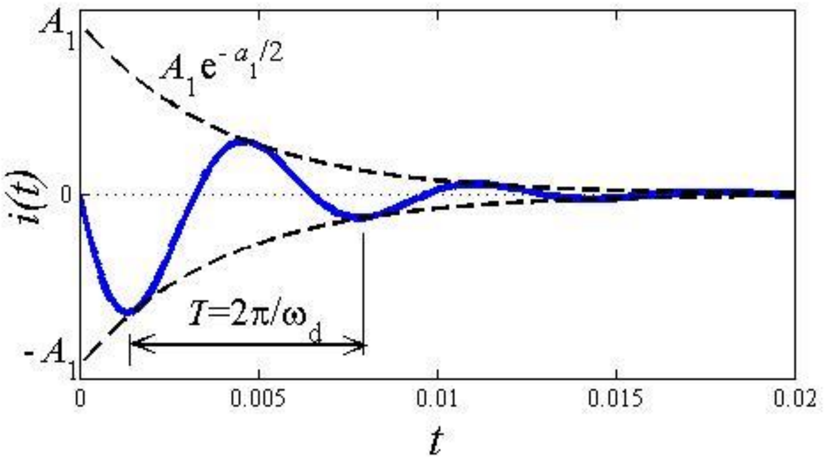
Type of solutions of the homogeneous equation

If $a_1^2 < 4a_0$, $(\xi < 1) \Rightarrow s_1 = s_2^*$

$$i(t) = A_1\,\mathrm{e}^{-\frac{a_1}{2}t}\sin\left(\frac{\sqrt{4a_0 - a_1^2}}{2}t + \phi\right):$$ **Underdamped**

here $\omega_\mathrm{d} := \dfrac{\sqrt{4a_0 - a_1^2}}{2}$ is the **damped natual frequency**

$$\left.\begin{array}{l} i(0) = A_1 \sin\phi = 0 \Rightarrow \phi = 0, \pm\pi, \pm 2\pi, \ldots \\ v_C(0) = 0 - L\left.\dfrac{di(t)}{dt}\right|_{t=0} = E_g \Rightarrow A_1 = \dfrac{-2E_g}{L\sqrt{4a_0 - a_1^2}} \end{array}\right\} \Rightarrow i(t) = \frac{-2E_g}{L\sqrt{4a_0 - a_1^2}}\,\mathrm{e}^{-\frac{a_1}{2}t}\sin\left(\frac{\sqrt{4a_0 - a_1^2}}{2}t\right).$$



The oscillation is a consequence of the energy exchange between *C* and *L*. First it moves from *C* to *L*, on the way some energy is dissipated by *R*. Once the remaining energy is stored in *L* it moves back to *C* dissipating again some energy in *R* and so on until all the energy is dissipated by *R*.

Type of solutions of the homogeneous equation

$$\text{If } a_1^2 = 4a_0, (\xi = 1) \Rightarrow s_1 = s_2 = -\frac{a_1}{2} = -\frac{R}{2L} = -\xi\omega_n$$

$$i(t) = \left(A_1 + A_2 t\right)\mathrm{e}^{s_1 t} :$$ **Critically damped**

$$\left.\begin{aligned} & i(0) = A_1 = 0 \Rightarrow i(t) = A_2 t\,\mathrm{e}^{s_1 t} \\ & v_C(0) = 0 - L\left.\frac{di(t)}{dt}\right|_{t=0} = E_g \Rightarrow A_2 = \frac{-E_g}{L} \end{aligned}\right\} \Rightarrow i(t) = \frac{-E_g}{L} t\,\mathrm{e}^{-\frac{a_1}{2}t}.$$

$$\text{If } a_1 = 0, (\xi = 0)$$

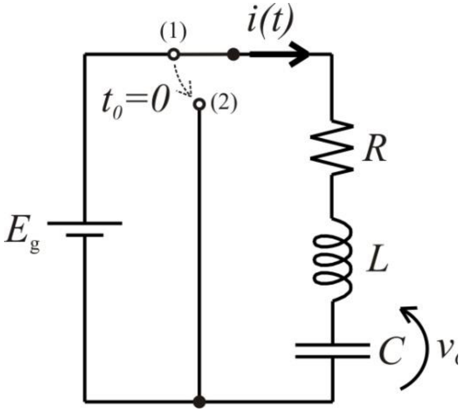
$$i(t) = A_1 \sin\left(\sqrt{a_0}\,t + \phi\right):$$ **Undamped**

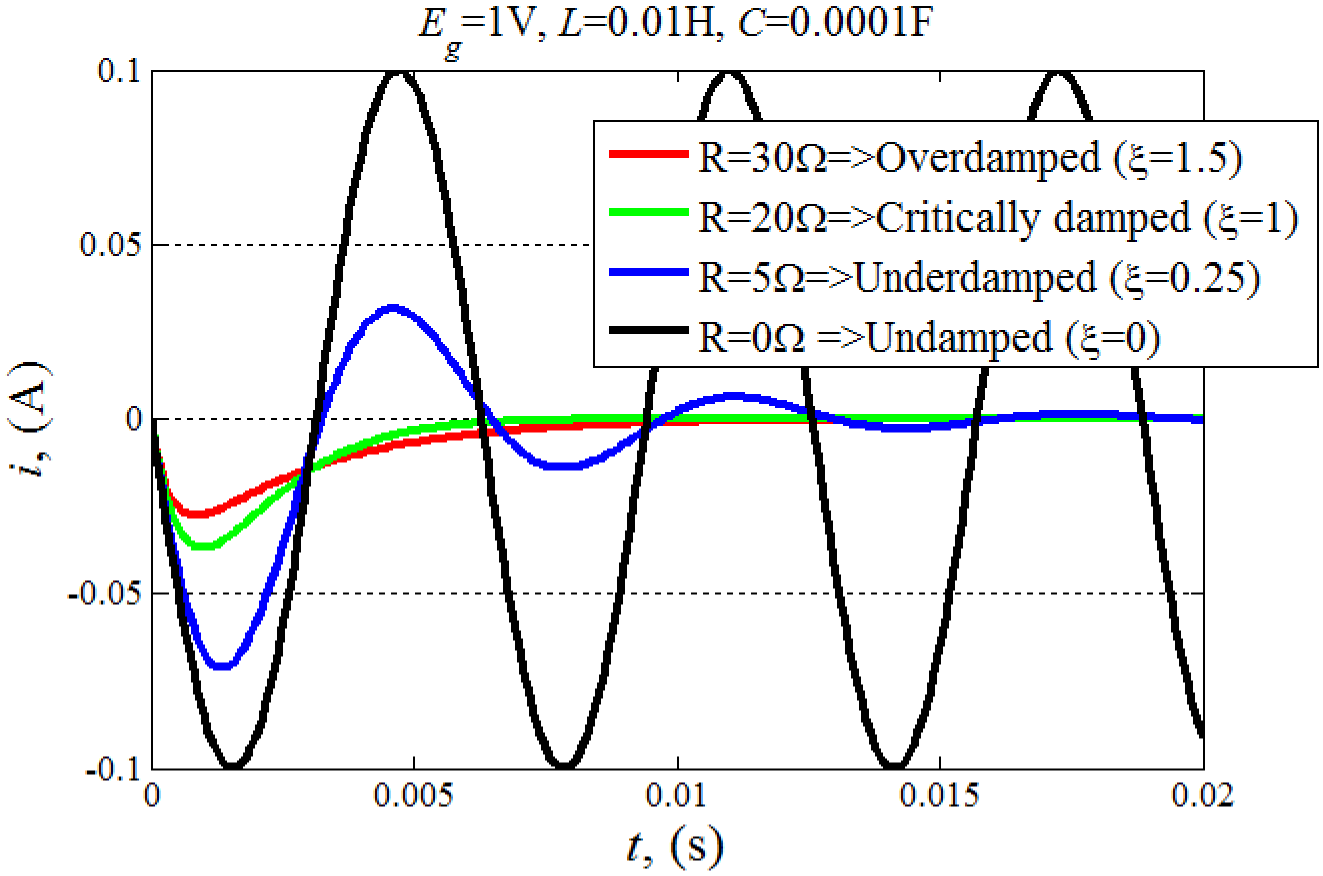
undamped natual frequency

$$\left.\begin{aligned} & i(0) = A_1 \sin\phi = 0 \Rightarrow \phi = 0, \pm\pi, \pm 2\pi, \ldots \\ & v_C(0) = 0 - L\left.\frac{di(t)}{dt}\right|_{t=0} = E_g \Rightarrow A_1 = \frac{-E_g}{L\sqrt{a_0}} \end{aligned}\right\} \Rightarrow i(t) = -E_g\sqrt{\frac{C}{L}}\sin\left(\frac{1}{\sqrt{LC}}t\right).$$

Type of solutions of the homogeneous equation

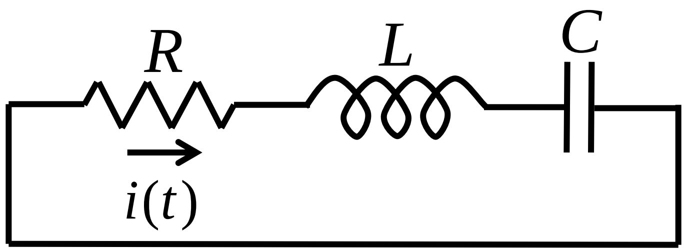
Discharge of the capacitor





Cause of the transient respinse

- RLC-serie. The resulting damping ratio is:

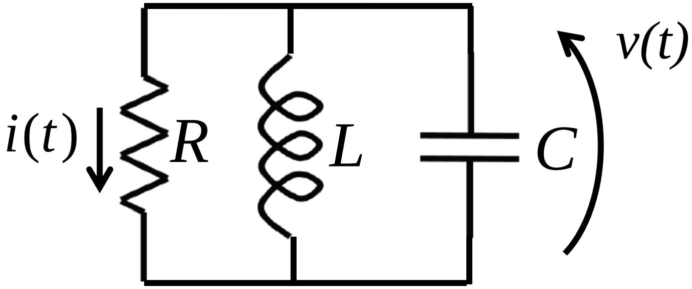


$$\xi = \frac{R}{2\omega_n L}$$

Is proportional to *R* because the energy dissipated in *R* increases with *R:*

$$p_R(t) = Ri^2(t)$$

- RLC-parallel. The resulting damping ratio is:



$$\xi = \frac{1}{2R\omega_n C}$$

Is inversely proportional to *R* since the energy dissipated in *R* decreases with *R:*

$$p_R(t) = Ri_R^2(t) = \frac{v^2(t)}{R}$$

Transient circuit's analysis using Laplace transforms

- By using Laplace transforms the circuits can be solved much easily:
  - No differential equation has to be obtained
  - We will solve algebraic instead of differential equations
  - No need to perform the tedious operations to calculate the constants (A_1,A_2,…) of the solution

Laplace transform ($\mathcal{L}$)

- The solutions are superposition's of exponential decreasing functions starting from the initial instant (*t*=0)

$$y(t) = \sum_n A_n \, \mathrm{e}^{s_n t}$$

- The Laplace transform ($\mathcal{L}$) allows to transform the differential equation into an algebraic equation with coefficients $A(s)$

$$A(s) = \mathcal{L}\left[y(t)\right] = \int_0^\infty y(t) \, \mathrm{e}^{-st} \, dt$$

Some properties of $\mathcal{L}$

Ohm's and Kirchhoff law's are still valid in $\mathcal{L}$-domain

- $\mathcal{L}$ is lineal, $F(s)= \mathcal{L}[f(t)]$

$$f_3(t) = af_1(t) + bf_2(t) \Leftrightarrow F_3(s) = aF_1(s) + bF_2(s)$$

- $\mathcal{L}$ of a derivation

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

Differential equation are transformed into algebraic equations

- Time translation

$$\mathcal{L}[f(t-t_0)] = \mathrm{e}^{-st_0}\, \mathcal{L}[f(t)], \quad \text{if } f(t<t_0)=0$$

- Translation *in* s domain

$$F(s-a) = \mathcal{L}\left[\mathrm{e}^{ta}\, f(t)\right]$$

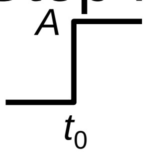
- Theorem of the final and initial value

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left(sF(s)\right),$$

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty}\left(sF(s)\right).$$

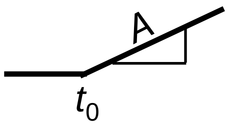
$\mathcal{L}$ of some functions

- Step function displaced in time



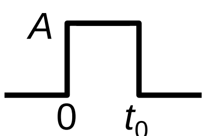
$$\mathcal{L}[Au(t-t_0)] = A\mathrm{e}^{-t_0 s}\,\mathcal{L}[u(t)] = \frac{A}{s}\mathrm{e}^{-t_0 s}$$

- Slope



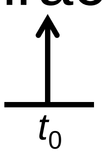
$$\mathcal{L}[A(t-t_0)u(t-t_0)] = A\mathrm{e}^{-t_0 s}\,\mathcal{L}[tu(t)] = \frac{A}{s^2}\mathrm{e}^{-t_0 s}$$

- Rec function



$$\mathcal{L}[A(u(t)-u(t-t_0))] = \frac{A}{s} - A\mathrm{e}^{-t_0 s}\,\mathcal{L}[u(t)] = \frac{A}{s}\left(1-\mathrm{e}^{-t_0 s}\right)$$

- Dirac delta function



$$\mathcal{L}[\delta(t-t_0)] = \mathrm{e}^{-t_0 s}\,\mathcal{L}[\delta(t)] = 1\mathrm{e}^{-t_0 s}$$

- Periodic functions with period ***T***

$$f(t) = \sum_n g(t-nT), \quad g(t) = f(t)\left(u(t)-u(t-T)\right)$$

$$\mathcal{L}[f(t)] = \sum_n \mathcal{L}[g(t)]\mathrm{e}^{nTs} = \frac{\mathcal{L}[g(t)]}{1-\mathrm{e}^{Ts}}$$

$\mathcal{L}$ of some functions

- Exponential

$$\mathcal{L}\left[e^{-at}\right]=\frac{1}{s+a}$$

- Sine

$$\mathcal{L}[\sin at]=\frac{a}{s^2+a^2}$$

- Cosine

$$\mathcal{L}[\cos at]=\frac{s}{s^2+a^2}$$

- In practice we will use a table with the most common inverse Laplace transforms ($\mathcal{L}^{-1}$) used for the resolution of the proposed problems

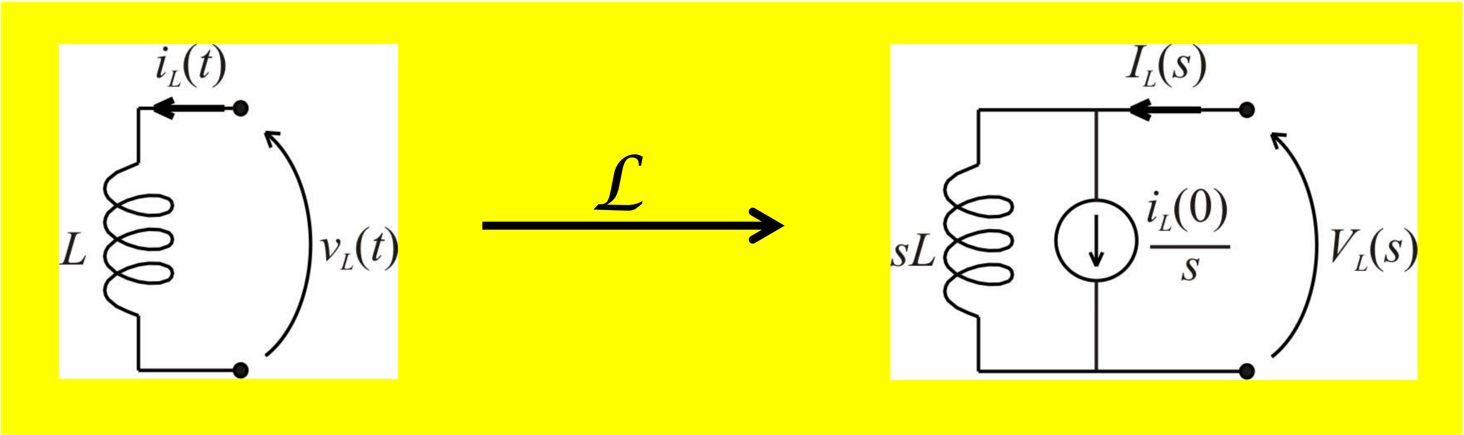
Resolution using Laplace

- The circuits will be solved in the Laplace domain:
  - Draw the circuit in the transformed domain, for this the initial conditions are deeded: $i_L(0), v_C(0)$.
  - The transformed circuit is then solved using the known methods, thus, by applying the Kirchhoff laws to the transformed currents and voltages: $I(s), V(s)$.(*)
  - Ones you know the Laplace transformed voltage or current, the inverse Laplace transform is applied to get the currents and voltages in the time domain

$$i(t) = \mathcal{L}^{-1}[I(s)], \quad v(t) = \mathcal{L}^{-1}[V(s)]$$

(*): Convention: Laplace transformed variables in capital letters

$\mathcal{L}$ transform of the inductor

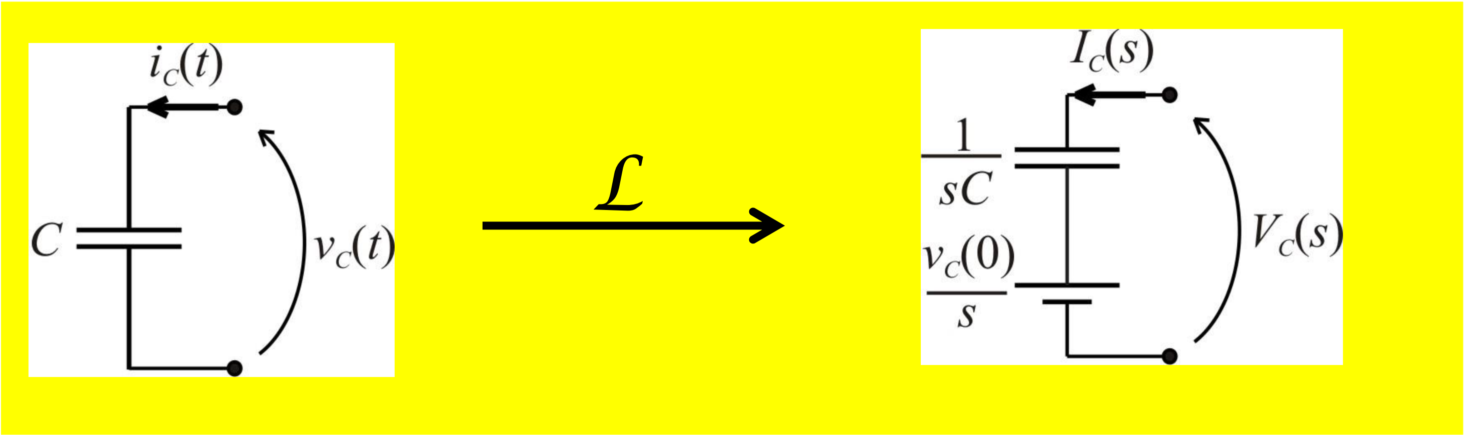


$$I_L(s) = \mathcal{L}[i_L(t)],$$

$$V_L(s) = \mathcal{L}[v_L(t)],$$

$$V_L(s) = \int_0^{\infty} v_L(t)e^{-st}dt = L\int_0^{\infty} \frac{di_L(t)}{dt}e^{-st}dt = Ls\left(I_L(s) - \frac{i_L(0)}{s}\right).$$

$\mathcal{L}$ transform of the capacitor



$$I_C(s) = \mathcal{L}[i_C(t)],$$

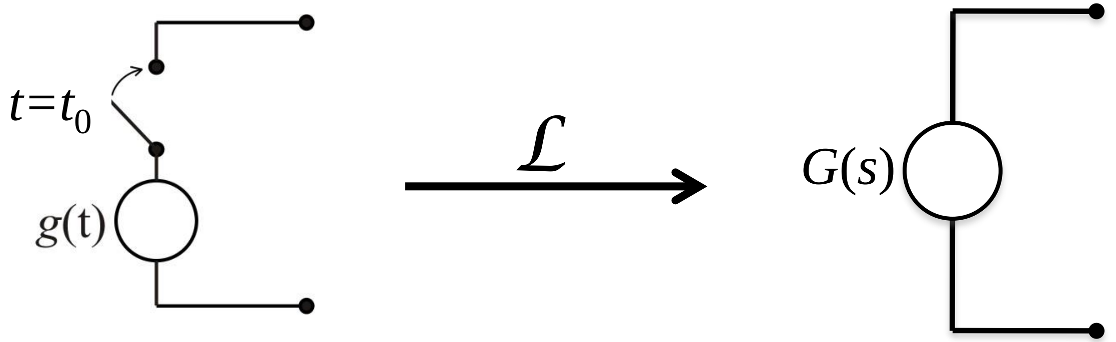
$$V_C(s) = \mathcal{L}[v_C(t)],$$

$$V_C(s) = \int_0^\infty v_C(t) e^{-st} dt = \frac{1}{C}\int_0^\infty q(t) e^{-st} dt = \frac{1}{C}\left(\frac{q(0)}{s} + \frac{1}{s} I_C(s)\right)$$

$$= \frac{1}{sC} I_C(s) + \frac{v_C(0)}{s}.$$

$\mathcal{L}$ transform of the generator

- After the switching:



For example: $\mathcal{L}[E]=\frac{E}{s},\ \mathcal{L}[\sin at]=\frac{a}{s^2+a^2},\ \mathcal{L}[\cos at]=\frac{s}{s^2+a^2}$

- If the switching happens at $\boldsymbol{t_0 \neq 0}$, perform a time translation by defining: $\boldsymbol{t'=t-t_0}$. This has to be taken into account by performing the inverse transform.

Initial conditions

- The initial condition we need are:
  - The currents trough each of the coils just after the switching takes place: $i_L(t_0^+)$
  - The voltages at the terminals of the capacitors after the switching takes place: $v_C(t_0^+)$
- If they are not known, then they have to be calculated by a previous (before the switching) resolution of the circuit,
  - Remember: $i_L(t_0^+)=i_L(t_0^-)$, $v_C(t_0^+)=v_C(t_0^-)$

Resolution in the $\mathcal{L}$-domain

- The transformed circuit is solved by applying the mesh or nod methods of the transformed voltages or the currents which depend on the variable *s*.
- The following expression for the voltage or current has to be obtained:

$$Y(s) \propto \frac{f(s)}{s^2 + a_1 s + a_0}$$

where the denominator is the characteristic equation:

$$s^2 + a_1 s + a_0 = 0 \Rightarrow s_{1,2} = \frac{1}{2}\left(-a_1 \pm \sqrt{a_1^2 - 4a_0}\right)$$

From which the roots (s_1 and s_2) are obtained, these allows:

- Predict the kind of solution
- Find the inverse in the Laplace transform table

Predicting of the kind of solution

- If the roods are real and different

$$Y(s) \propto \frac{f(s)}{s^2 + a_1 s + a_0} = \frac{f(s)}{(s-s_1)(s-s_2)} \xrightarrow{\mathcal{L}^{-1}} y(t) \propto A_1 \mathrm{e}^{s_1 t} + A_1 \mathrm{e}^{s_2 t}$$

- For equal and real roots

$$Y(s) \propto \frac{f(s)}{s^2 + a_1 s + a_0} = \frac{f(s)}{(s-s_1)^2} \xrightarrow{\mathcal{L}^{-1}} y(t) \propto (A_1 + A_2 t)\mathrm{e}^{s_1 t}$$

- For complex conjugated roots: $s_{1,2}=p \pm \mathrm{j}q$

$$Y(s) \propto \frac{f(s)}{s^2 + a_1 s + a_0} = \frac{f(s)}{(s-p)^2 + q^2} \xrightarrow{\mathcal{L}^{-1}} y(t) \propto Ae^{pt} \sin(qt + \phi)$$

- Imaginary roots: $s_{1,2}=\pm \mathrm{j}q$

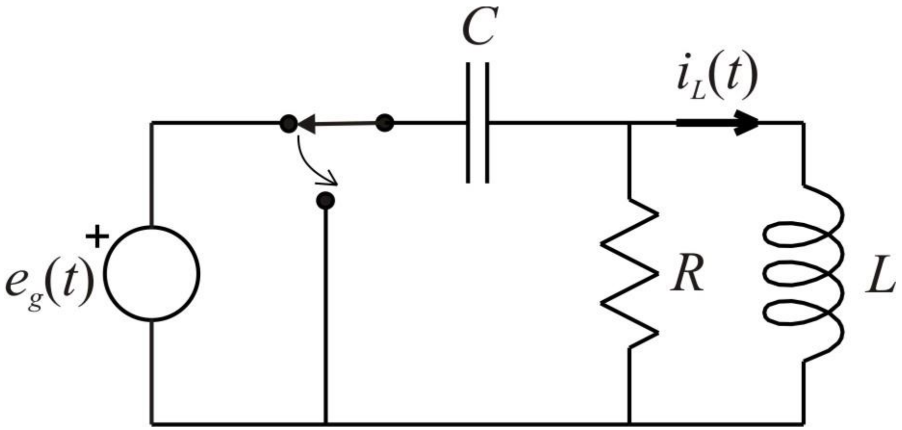
$$Y(s) \propto \frac{f(s)}{s^2 + a_1 s + a_0} = \frac{f(s)}{s^2 + q^2} \xrightarrow{\mathcal{L}^{-1}} y(t) \propto A\sin(qt + \phi)$$

Inverse Laplace transform ($\mathcal{L}^{-1}$)

- The roots of the characteristic equation allows:
  - To know beforehand the kind of solution
  - It makes easier to find the inverse Laplace transform in the inverse Laplace transform table.
- Do not forget: if there was a time shifting, substitute *t'* by $t\text{-}t_0$ in the inverse transform.
- The obtained solution is defined for a time interval after the switching.
- **Check the initial condition.**

Example 1

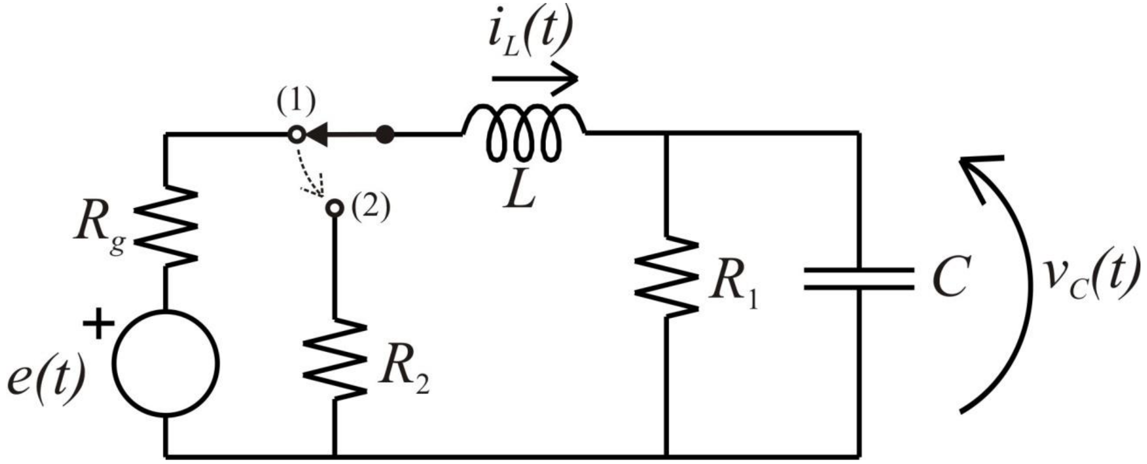
- In the circuit of the figure, the switcher is in position (1) since $t = -\infty$. At $t = \pi/2$ the switcher changes to position (2). Obtain the temporal evolution of $i_L(t)$.



Data: $e_g(t)$=2cos2t V, R=2Ω, L=1H, C=0.5F.

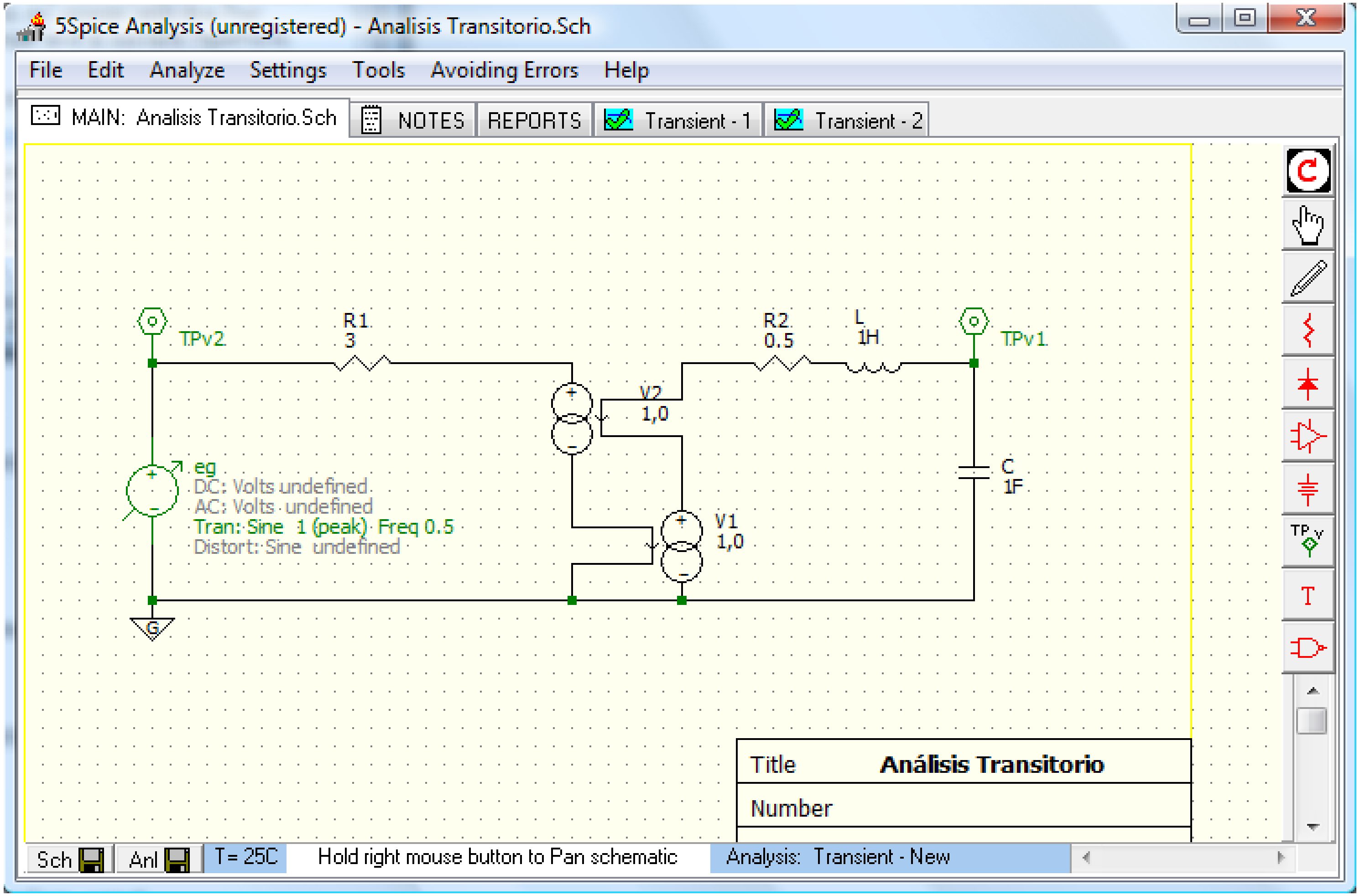
Example 2

- In the circuit of the figure, the switcher is in position (1) since $t = -\infty$. At $t = 0$ the switcher switches to position (2). Obtain the temporal evolution of $v_C(t)$.



Data: $e(t)$=10V, R_g=R_1=1Ω, R_2=2Ω, L=1H, C=2F.

Simulation with 5Spice



Simulación con 5Spice