U2. Two-port Networks or Quadripoles

Circuit Analysis, Grado TIC

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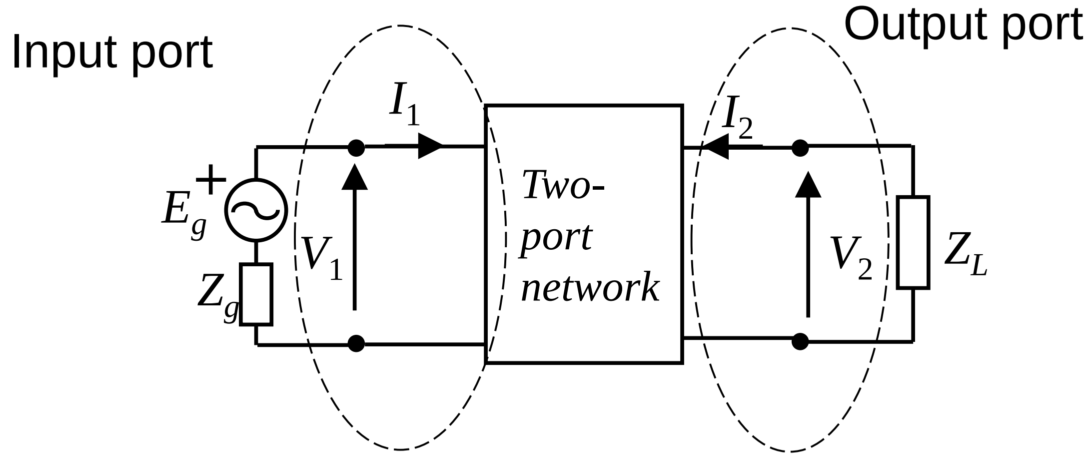
Index

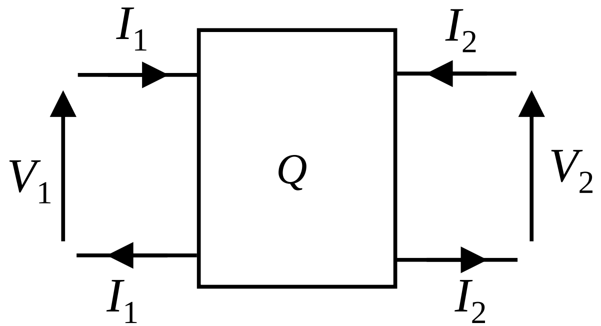
Two-port network or quadripole (Q)

- It is a external characterization of a *linear* and *passive* circuit that allows to know how it will interact with other elements
- It has two ports to be connected to other elements: the input and the output port
- It is then a "black box" that is usually connected between a generator (E_g,Z_g) and a load element (Z_L)



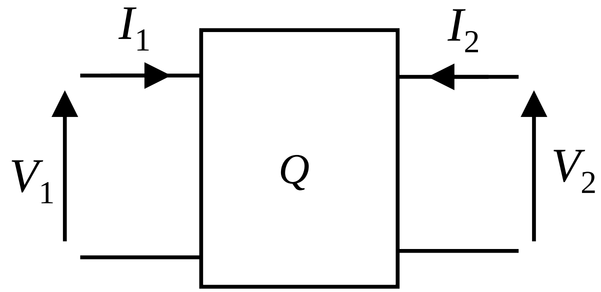
Two-port network or quadripole (Q)

- **Q is characterized with 4 parameters** that will allow us to directly relate I_1, I_2, V_1 and V_2
- Q's are characterized by:
  - Initial conditions are zero (when analyzing in Laplace domain no energy is stored before any connection)
  - It is **passive** (it does not have independent sources)
  - At **each port** the current that enters is the same as the current that exits (port condition)



Two-port network or quadripoles (Q)

- The voltages and currents at the input and output ports can be related each other with different family of parameters: “z”, “y”, “h”,…



$$\text{"}\mathbf{z}\text{"}: \begin{cases} V_1 = z_{11} I_1 + z_{12} I_2 \\ V_2 = z_{21} I_1 + z_{22} I_2 \end{cases}, \text{"}\mathbf{y}\text{"}: \begin{cases} I_1 = y_{11} V_1 + y_{12} V_2 \\ I_2 = y_{21} V_1 + y_{22} V_2 \end{cases}, \text{"}\mathbf{h}\text{"}: \begin{cases} V_1 = h_{11} I_1 + h_{12} V_2 \\ I_2 = h_{21} I_1 + h_{22} V_2 \end{cases} ,\dots$$

You don’t need to know the internal structure of the network, it suffices with the parameters to study the behavior of Q

Recirpocal and Symmetrical Two-port networks

- Q is **reciprocal** when Q is “flipped over”(*) and the relationship between the voltage (or current) applied at one port and the current (or voltage) measured at the other port does not change
- Q is **symmetrical** when Q is flipped over and there is no change in any current and voltage at the input and output gate

(*): the input port is changed with the output port

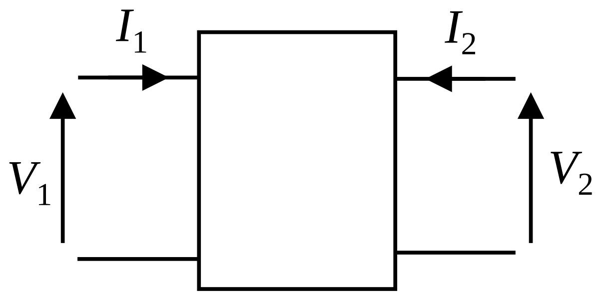
Recirpocal and Symmetrical Two-port networks

- Any symmetrical Q is also reciprocal
- A special case of reciprocal Q’s are the one made of linear elements only (L, R, C), also known as bilateral Q.
- For symmetrical Q the input- and output impedances will be the same if Q is connected to the same load (at the output- and input port respectively).

Objectives

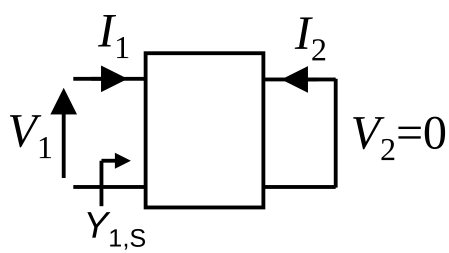
- Definition and determination of the Q parameters of the different families from measurements at their port’s
- Analyze circuits with Q’s whose parameters are known.
- The Q parameters are in general defined in the Laplace transformed domain, but we will consider SSS regimen (where s = jw))

Admittance parameters "y"



$$\begin{cases} I_1 = y_{11}V_1 + y_{12}V_2 \\ I_2 = y_{21}V_1 + y_{22}V_2 \end{cases}$$

Short-circuit at the output port:

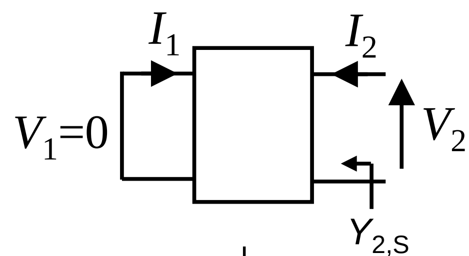


$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = Y_{1,S} [=] \Omega^{-1}$$

$$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

Short-circuit transfer admittance i→o

Short-circuit at the input port:



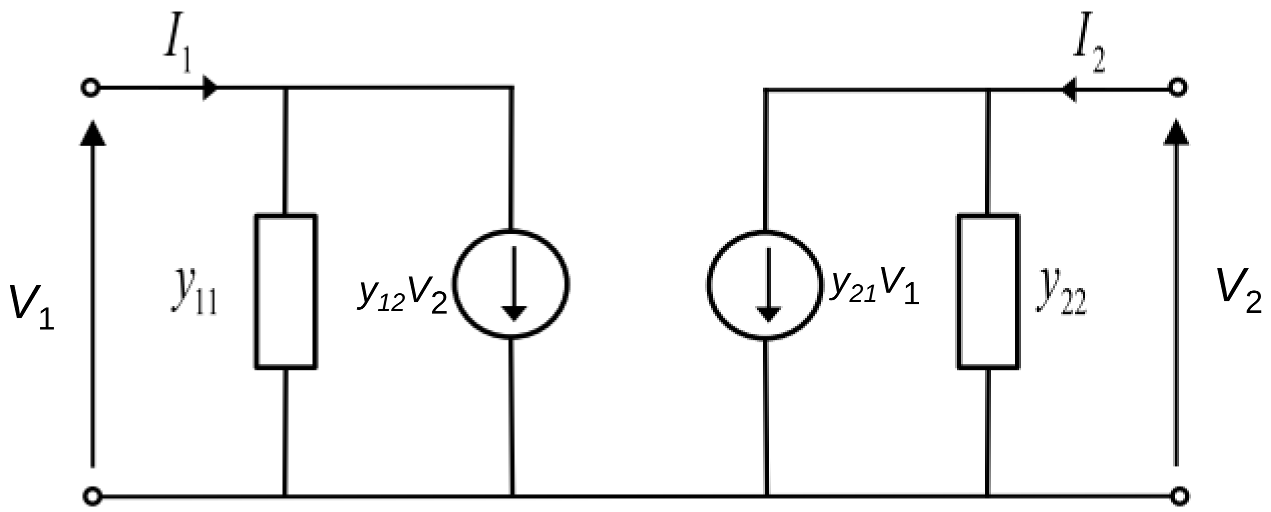
$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

S.c. transfer admittance o→i

$$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = Y_{2,S}$$

Admittance parameters "y"

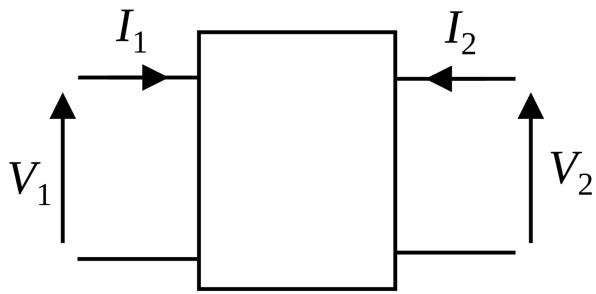
- Equivalent circuit



$$\begin{cases} I_1 = y_{11}V_1 + y_{12}V_2 \\ I_2 = y_{21}V_1 + y_{22}V_2 \end{cases}$$

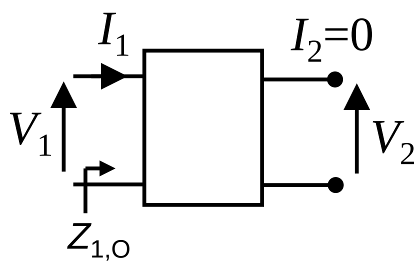
Q is reciprocal $\Rightarrow y_{12}=y_{21}$
Q is symmetrical $\Rightarrow y_{11}=y_{22}$ & $y_{12}=y_{21}$

Impedance parameters "**z**"



$$\begin{cases} V_1 = z_{11} I_1 + z_{12} I_2 \\ V_2 = z_{21} I_1 + z_{22} I_2 \end{cases}$$

Open-circuit at the output port:

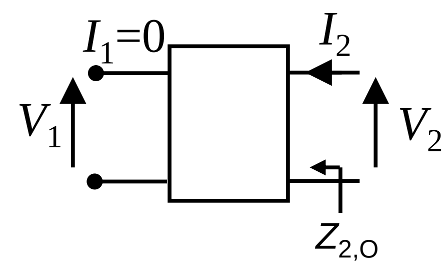


$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_{1,O} [=] \Omega$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

O.c. transfer impedance i→o

Open-circuit at the input port:



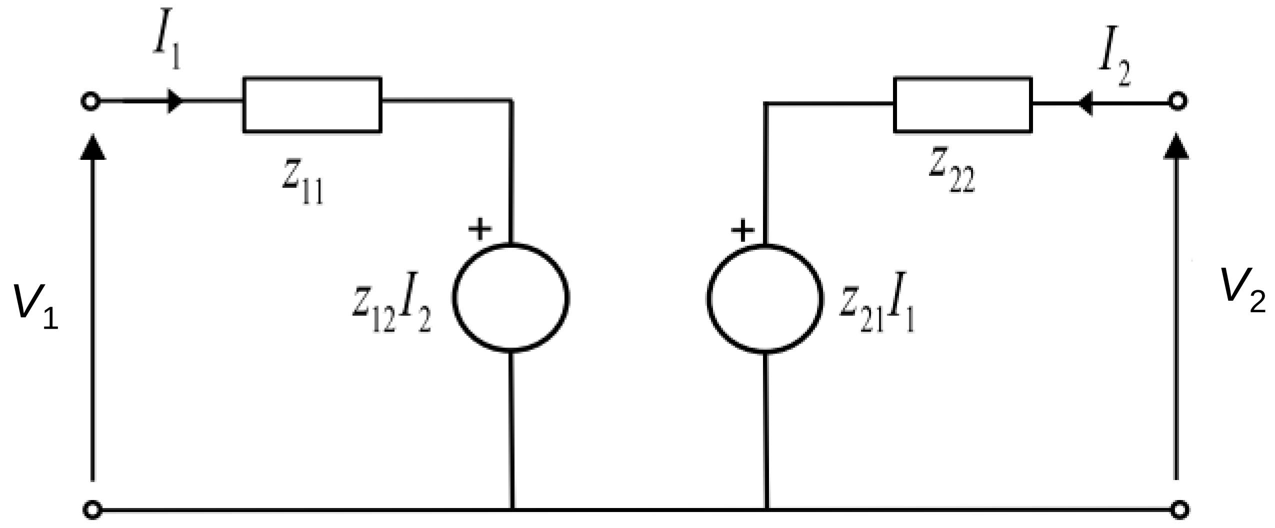
$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

O.c. transfer impedance o→i

$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = Z_{2,O}$$

Impedance parameters "**z**"

- Equivalent circuit

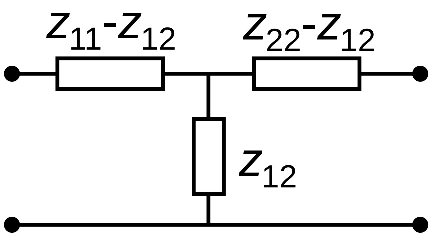


$$\begin{cases} V_1 = z_{11}I_1 + z_{12}I_2 \\ V_2 = z_{21}I_1 + z_{22}I_2 \end{cases}$$

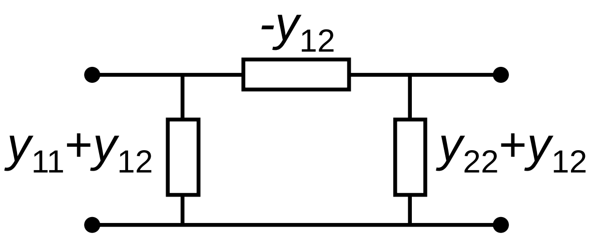
Q is reciprocal $\Rightarrow z_{12}=z_{21}$
Q is symmetrical $\Rightarrow z_{11}=z_{22}$ & $z_{12}=z_{21}$

Bilateral and symmetrical quadripoles

- Example of quadripoles characterized with z and y-parameters



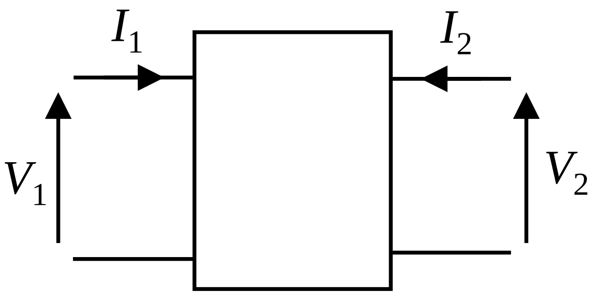
T-equivalent circuit



Π-equivalent circuit

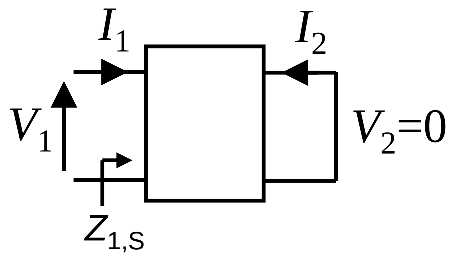
- Some application of z- and y-parameters
  - Impedance matching (optimize the power transference from source to load impedance)
  - Power distribution networks
  - Filter design (unit 4)

Hybrid parameters "h"



$$\begin{cases} V_1 = h_{11} I_1 + h_{12} V_2 \\ I_2 = h_{21} I_1 + h_{22} V_2 \end{cases}$$

Short-circuit at the output port:

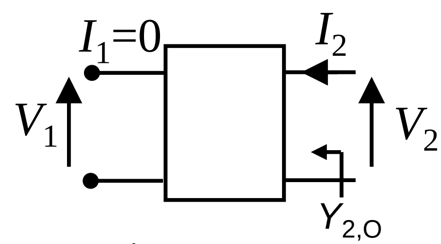


$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = Z_{1,S} [=] \Omega$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

S.c. forward (i→o) current gain

Open-circuit at the input port:



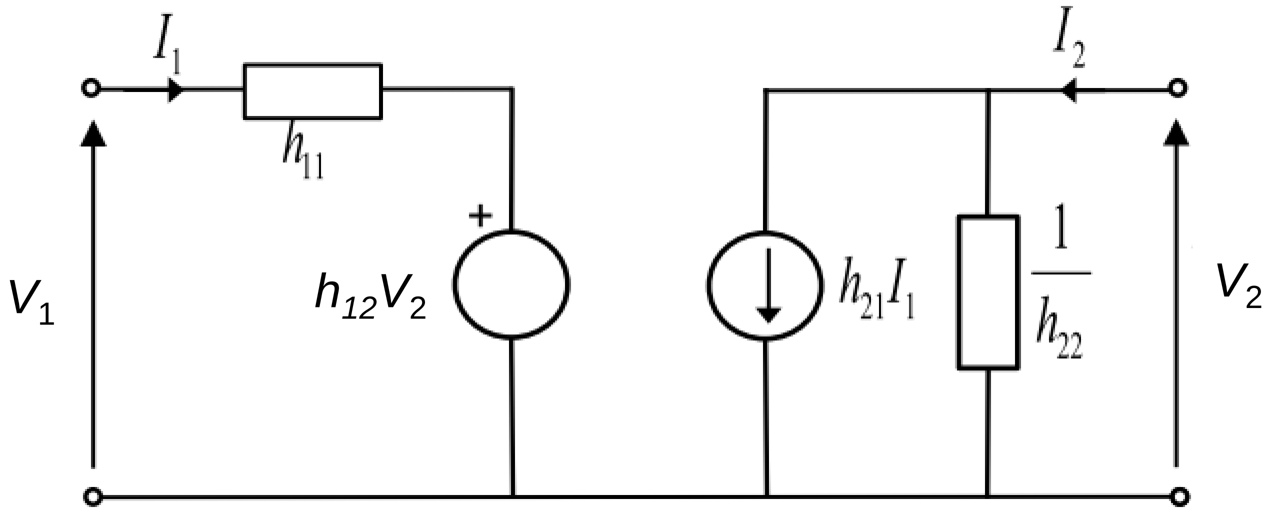
$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

O.c. reverse (o→i) voltage gain

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = Y_{2,O} [=] \Omega^{-1}$$

Hybrid parameters "**h**"

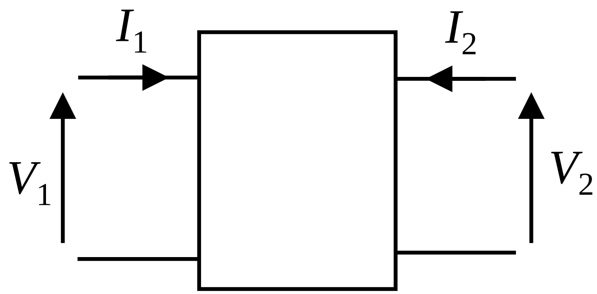
- Equivalent circuit



$$\begin{cases} V_1 = h_{11}I_1 + h_{12}V_2 \\ I_2 = h_{21}I_1 + h_{22}V_2 \end{cases}$$

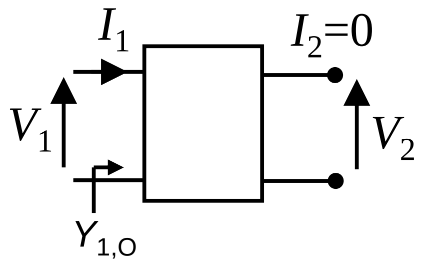
Q is reciprocal $\Rightarrow h_{12}=-h_{21}$
Q is symmetrical $\Rightarrow$ det(**h**)=$h_{11}h_{22}-h_{12}h_{21}=1$ & $h_{12}=-h_{21}$

Inverse hybrid parameters "**g**"



$$\begin{cases} I_1 = g_{11}V_1 + g_{12}I_2 \\ V_2 = g_{21}V_1 + g_{22}I_2 \end{cases}$$

Open-circuit at the output port:

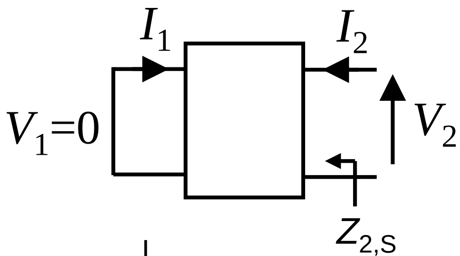


$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} = Y_{1,O} [=] \Omega^{-1}$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0}$$

O.c. forward (i→o) voltage gain

Short-circuit at the input port:



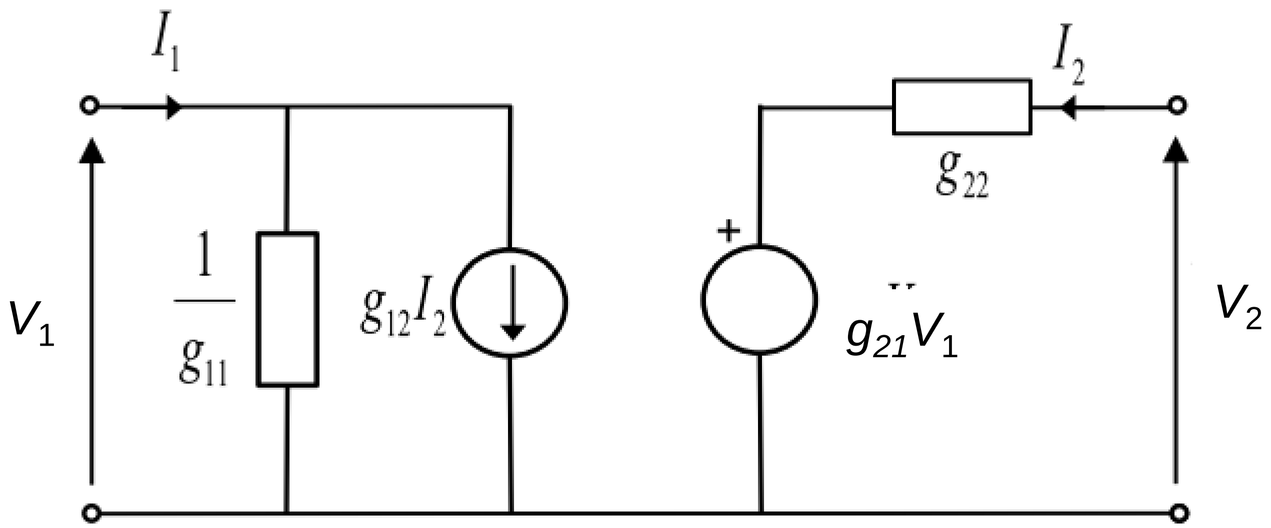
$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0}$$

S.c. reverse (o→i) current gain

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} = Z_{2,S} [=] \Omega$$

Inverse hybrid parameters "**g**"

- Equivalent circuit

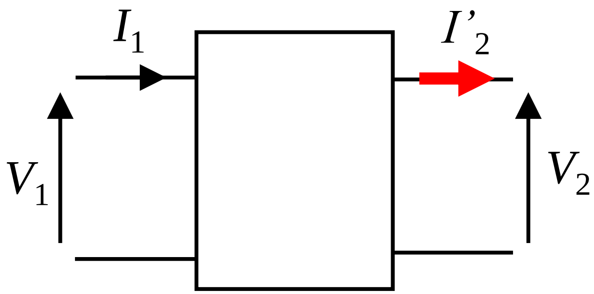


$$\begin{cases} I_1 = g_{11}V_1 + g_{12}I_2 \\ V_2 = g_{21}V_1 + g_{22}I_2 \end{cases}$$

Q is reciprocal $\Rightarrow g_{12}=-g_{21},$
Q is symmetrical $\Rightarrow \det(\mathbf{g})=g_{11}g_{22}-g_{12}g_{21}=1$ & $g_{12}=-g_{21}$

g- and h-parameters are used for modeling transformers and transistors

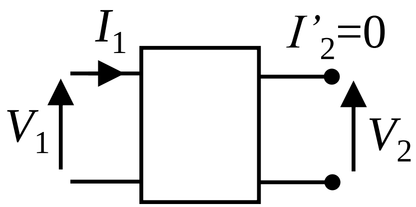
Transmission parameters "**T**" (o "*ABCD*")



$$\begin{cases} V_1 = AV_2 + BI'_2 \\ I_1 = CV_2 + DI'_2 \end{cases}, \quad \mathbf{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$(I'_2 = -I_2)$

Open-circuit at the output port:



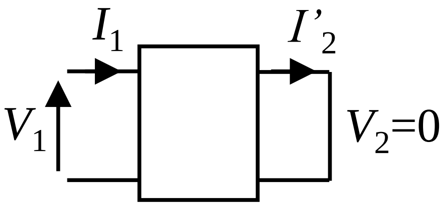
$$A = \left.\frac{V_1}{V_2}\right|_{I'_2=0}$$

O.c. reverse (o→i) voltage gain.

$$C = \left.\frac{I_1}{V_2}\right|_{I'_2=0}$$

O.c. transfer admittance (o→i)

Short-circuit at the output port :



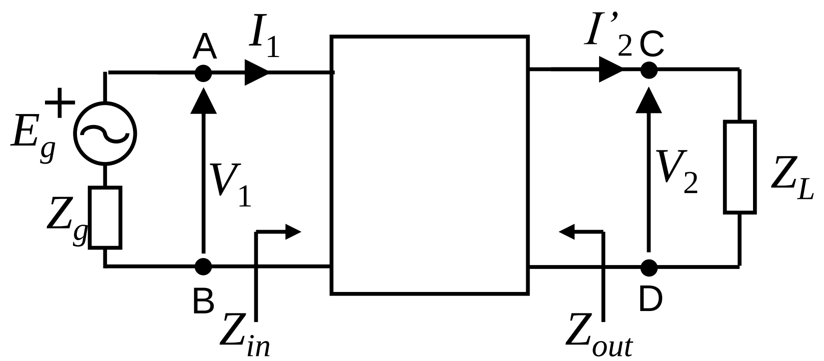
$$B = \left.\frac{V_1}{I'_2}\right|_{V_2=0}$$

S.c. transfer impedance o→i

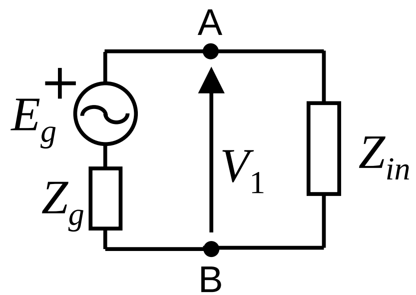
$$D = \left.\frac{I_1}{I'_2}\right|_{V_1=0}$$

S.c. reverse (o→i) current gain

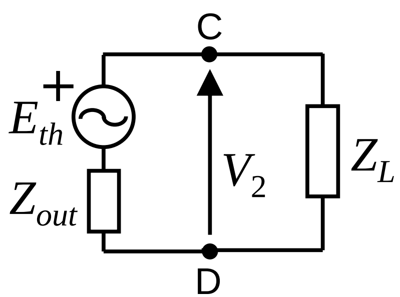
Thevenin and equivalent impedance seen trough Q using “T” parameters



Equivalent circuit:



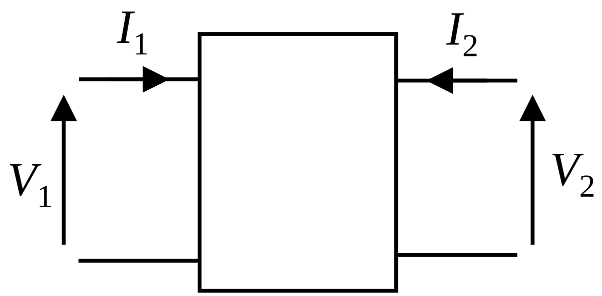
$$Z_{in} = \frac{AZ_L + B}{CZ_L + D}$$



$$Z_{out} = \frac{DZ_g + B}{CZ_g + A}, \quad E_{th} = \frac{E_g}{CZ_g + A}$$

(Note: if A=D and Z_g=Z_L then Z_{in}=Z_{out})

Summary of the Q parameters



$$\begin{cases} V_1 = AV_2 - BI_2 \\ I_1 = CV_2 - DI_2 \end{cases}$$

$$\begin{cases} V_1 = z_{11}I_1 + z_{12}I_2 \\ V_2 = z_{21}I_1 + z_{22}I_2 \end{cases}$$

$$\begin{cases} I_1 = g_{11}V_1 + g_{12}I_2 \\ V_2 = g_{21}V_1 + g_{22}I_2 \end{cases}$$

$$\begin{cases} I_1 = y_{11}V_1 + y_{12}V_2 \\ I_2 = y_{21}V_1 + y_{22}V_2 \end{cases}$$

$$\begin{cases} V_1 = h_{11}I_1 + h_{12}V_2 \\ I_2 = h_{21}I_1 + h_{22}V_2 \end{cases}$$

There exist two-port network’s for which some family of parameters are not defined, for example, ideal transformer.

If Q is reciprocal and symmetrical

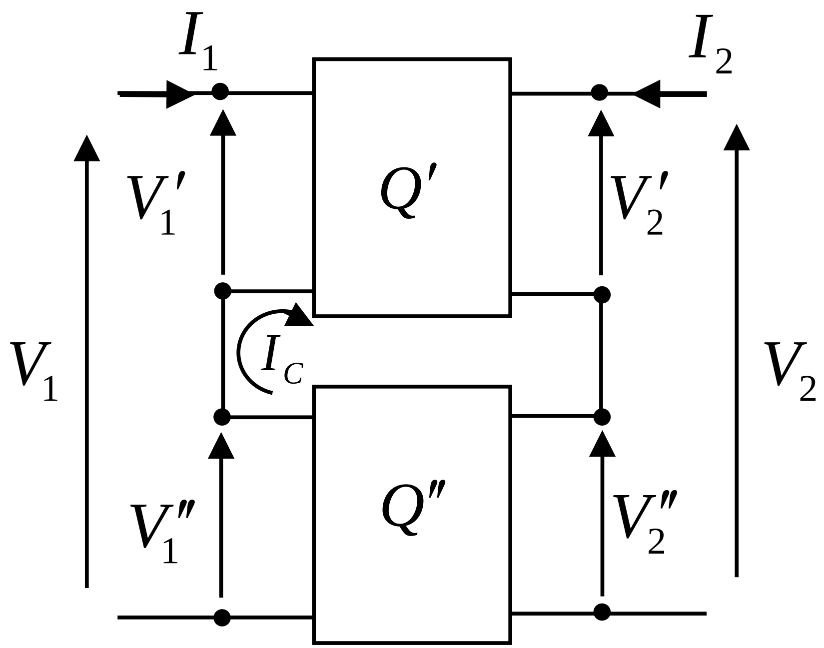
Family	Reciprocal only fulfill:	Symmetrical also fulfill
“z”	$z_{12}=z_{21}$	$z_{11}=z_{22}$
“y”	$y_{12}=y_{21}$	$y_{11}=y_{22}$
“h”	$h_{12}=-h_{21}$	$\lvert \mathbf{h} \rvert=1$
“g”	$g_{12}=-g_{21}$	$\lvert \mathbf{g} \rvert=1$
“T”	$\lvert \mathbf{T} \rvert=AD-BC=1$	$A=D$

Relations between the parameters

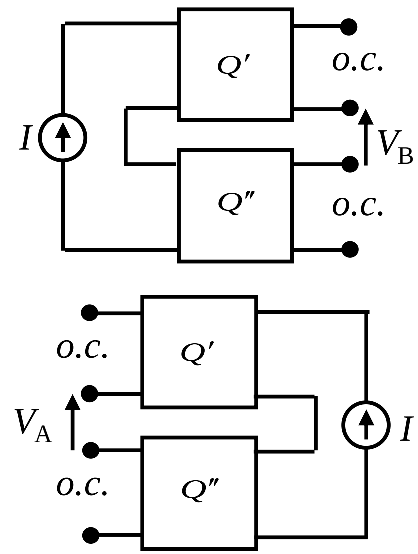
	"z"	"y"	"h"	"g"	"T"
"z"		$\begin{matrix} y_{22}/\lvert\mathbf{y}\rvert & -y_{12}/\lvert\mathbf{y}\rvert \\ -y_{21}/\lvert\mathbf{y}\rvert & y_{11}/\lvert\mathbf{y}\rvert \end{matrix}$	$\begin{matrix} \lvert\mathbf{h}\rvert/h_{22} & h_{12}/h_{22} \\ -h_{21}/h_{22} & 1/h_{22} \end{matrix}$	$\begin{matrix} 1/g_{11} & -g_{12}/g_{11} \\ g_{21}/g_{11} & \lvert\mathbf{g}\rvert/g_{11} \end{matrix}$	$\begin{matrix} A/C & \lvert\mathbf{T}\rvert/C \\ 1/C & D/C \end{matrix}$
"y"	$\begin{matrix} z_{22}/\lvert\mathbf{z}\rvert & -z_{12}/\lvert\mathbf{z}\rvert \\ -z_{21}/\lvert\mathbf{z}\rvert & z_{11}/\lvert\mathbf{z}\rvert \end{matrix}$		$\begin{matrix} 1/h_{11} & -h_{12}/h_{11} \\ h_{21}/h_{11} & \lvert\mathbf{h}\rvert/h_{11} \end{matrix}$	$\begin{matrix} \lvert\mathbf{g}\rvert/g_{22} & g_{12}/g_{22} \\ -g_{21}/g_{22} & 1/g_{22} \end{matrix}$	$\begin{matrix} D/B & -\lvert\mathbf{T}\rvert/B \\ -1/B & A/B \end{matrix}$
"h"	$\begin{matrix} \lvert\mathbf{z}\rvert/z_{22} & z_{12}/z_{22} \\ -z_{21}/z_{22} & 1/z_{22} \end{matrix}$	$\begin{matrix} 1/y_{11} & -y_{12}/y_{11} \\ y_{21}/y_{11} & \lvert\mathbf{y}\rvert/y_{11} \end{matrix}$		$\begin{matrix} g_{22}/\lvert\mathbf{g}\rvert & -g_{12}/\lvert\mathbf{g}\rvert \\ -g_{21}/\lvert\mathbf{g}\rvert & g_{11}/\lvert\mathbf{g}\rvert \end{matrix}$	$\begin{matrix} B/D & \lvert\mathbf{T}\rvert/D \\ -1/D & C/D \end{matrix}$
"g"	$\begin{matrix} 1/z_{11} & -z_{12}/z_{11} \\ z_{21}/z_{11} & \lvert\mathbf{z}\rvert/z_{11} \end{matrix}$	$\begin{matrix} \lvert\mathbf{y}\rvert/y_{22} & y_{12}/y_{22} \\ -y_{21}/y_{22} & 1/y_{22} \end{matrix}$	$\begin{matrix} h_{22}/\lvert\mathbf{h}\rvert & -h_{12}/\lvert\mathbf{h}\rvert \\ -h_{21}/\lvert\mathbf{h}\rvert & h_{11}/\lvert\mathbf{h}\rvert \end{matrix}$		$\begin{matrix} C/A & -\lvert\mathbf{T}\rvert/A \\ -1/A & B/A \end{matrix}$
"T"	$\begin{matrix} z_{11}/z_{21} & \lvert\mathbf{z}\rvert/z_{21} \\ 1/z_{21} & z_{22}/z_{21} \end{matrix}$	$\begin{matrix} -y_{22}/y_{21} & -1/y_{21} \\ -\lvert\mathbf{y}\rvert/y_{21} & -y_{11}/y_{21} \end{matrix}$	$\begin{matrix} -\lvert\mathbf{h}\rvert/h_{21} & -h_{11}/h_{21} \\ -h_{22}/h_{21} & -1/h_{21} \end{matrix}$	$\begin{matrix} 1/g_{21} & g_{22}/g_{21} \\ g_{11}/g_{21} & \lvert\mathbf{g}\rvert/g_{21} \end{matrix}$	

Association of Q's

Serial-serial



Brune Tests:



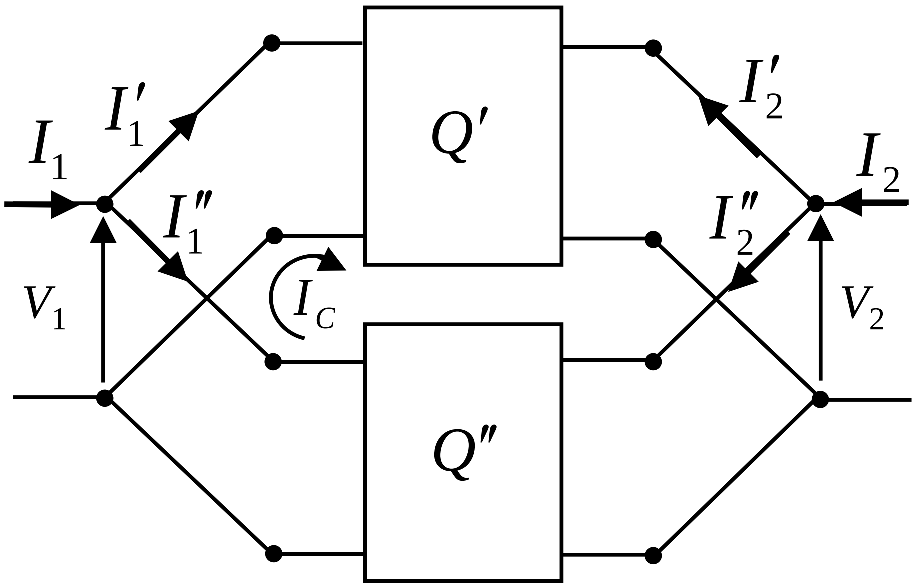
If V_A=0 and V_B=0

$\forall\, I \;\Rightarrow I_C$=0.

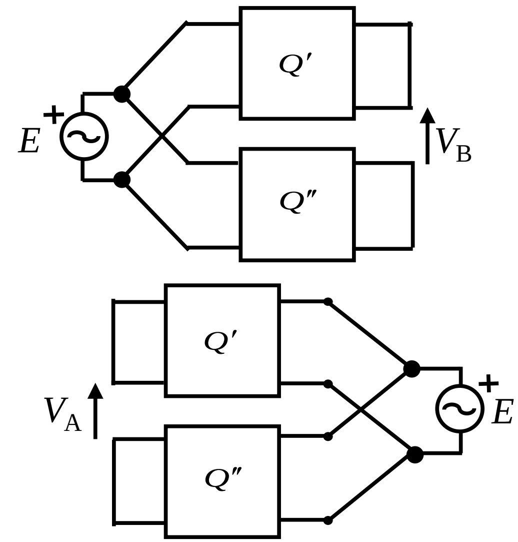
$$\text{If } I_C = 0 \Rightarrow \begin{cases} V_1 = V_1' + V_1'' =_1 \left(z_{11}' + z_{11}''\right) I_1 + \left(z_{12}' + z_{12}''\right) I_2 \\ V_2 = V_2' + V_2'' = \left(z_{21}' + z_{21}''\right) I_1 + \left(z_{22}' + z_{22}''\right) I_2 \end{cases}$$

Association of Q's

Parallel-parallel



Brune Tests:

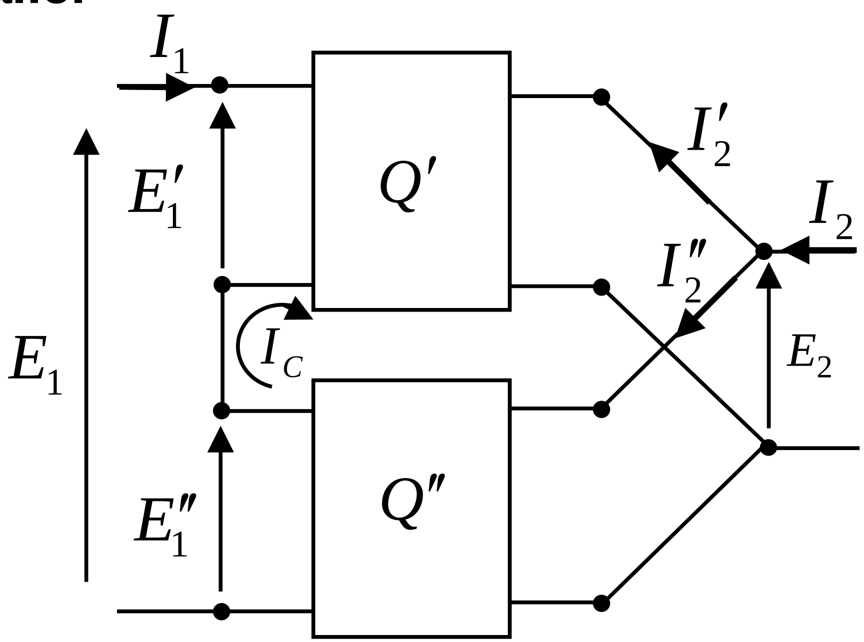


If V_A=0 and V_B=0
$\forall E \Rightarrow I_C$=0.

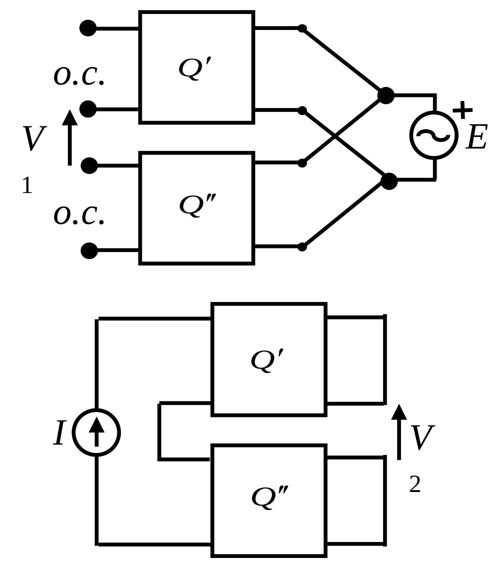
$$\text{if all } I_C = 0 \Rightarrow \begin{cases} I_1 = I_1' + I_1'' = (y_{11}' + y_{11}'')V_1 + (y_{12}' + y_{12}'')V_2 \\ I_2 = I_2' + I_2'' = (y_{21}' + y_{21}'')V_1 + (y_{22}' + y_{22}'')V_2 \end{cases}$$

Association of Q's

Serial-parallel



Brune Tests:

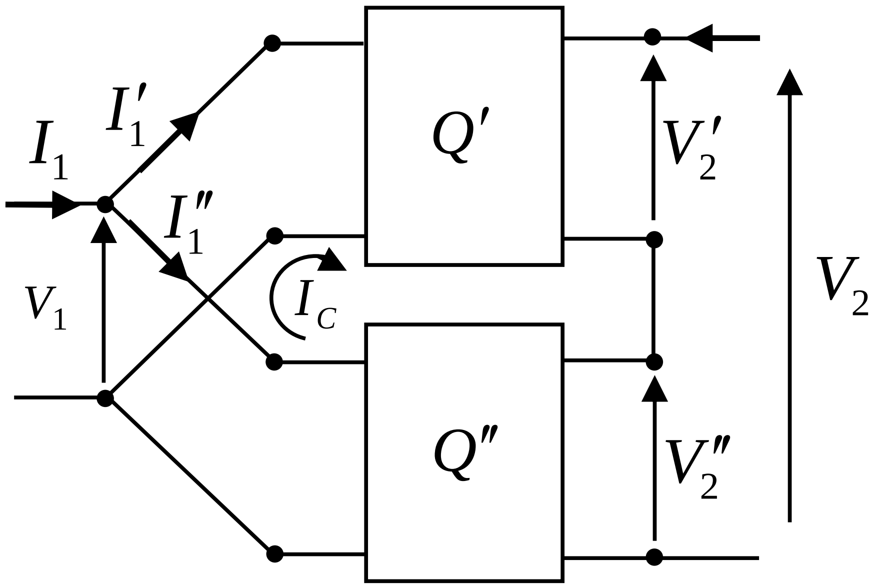


If V_1=0 $\forall E$ and V_2=0 $\forall I \Rightarrow I_C$=*0.*

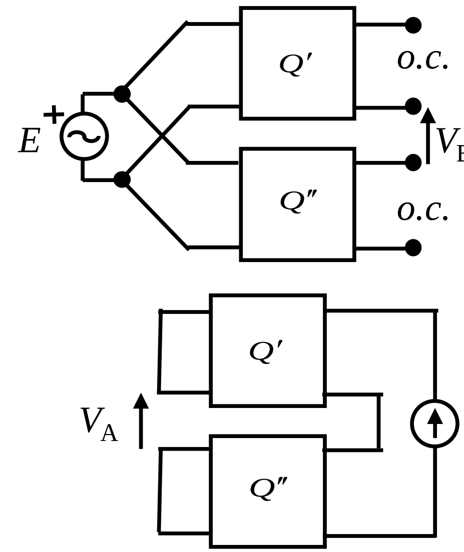
$$\text{if all } I_C = 0 \Rightarrow \begin{cases} E_1 = E'_1 + E''_1 =_1 \left(h'_{11} + h''_{11}\right) I_1 + \left(h'_{12} + h''_{12}\right) E_2 \\ I_2 = I'_2 + I''_2 = \left(h'_{21} + h''_{21}\right) I_1 + \left(h'_{22} + h''_{22}\right) E_2 \end{cases}$$

Association of Q's

Parallel-serial



Brune Tests:

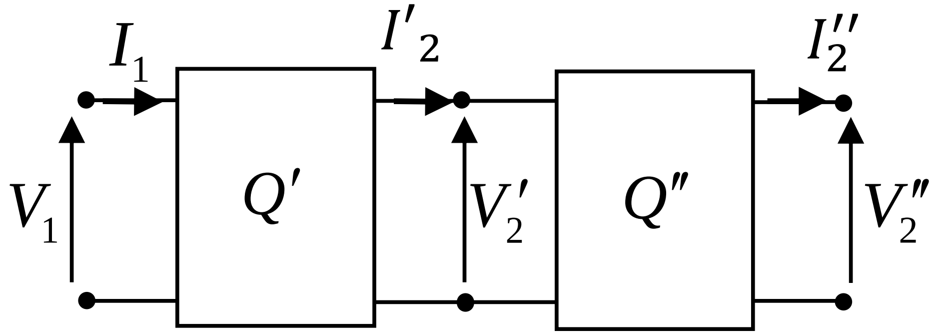


If V_B=0 $\forall E$ and V_A=0 $\forall I \Rightarrow I_C$=0.

$$\text{if all } I_C = 0 \Rightarrow \begin{cases} I_1 = I_1' + I_1'' = (g_{11}' + g_{11}'')V_1 + (g_{12}' + g_{12}'')I_2 \\ V_2 = V_2' + V_2'' = (g_{21}' + g_{21}'')V_1 + (g_{22}' + g_{22}'')I_2 \end{cases}$$

Association of Q's

Cascaded



$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} V'_2 \\ I'_2 \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix} \begin{pmatrix} V''_2 \\ I''_2 \end{pmatrix}$$

For *n*=1 to *N* Q's connected in cascade: $\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \prod_{n=1}^{N} \mathbf{T}_n \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$