# U4. Introduction to passive filters

Circuit Analysis, Grado en Ingeniería de Comunicaciones
Curso 2013-2014

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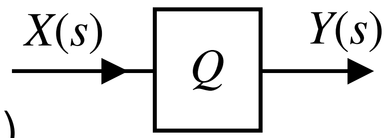
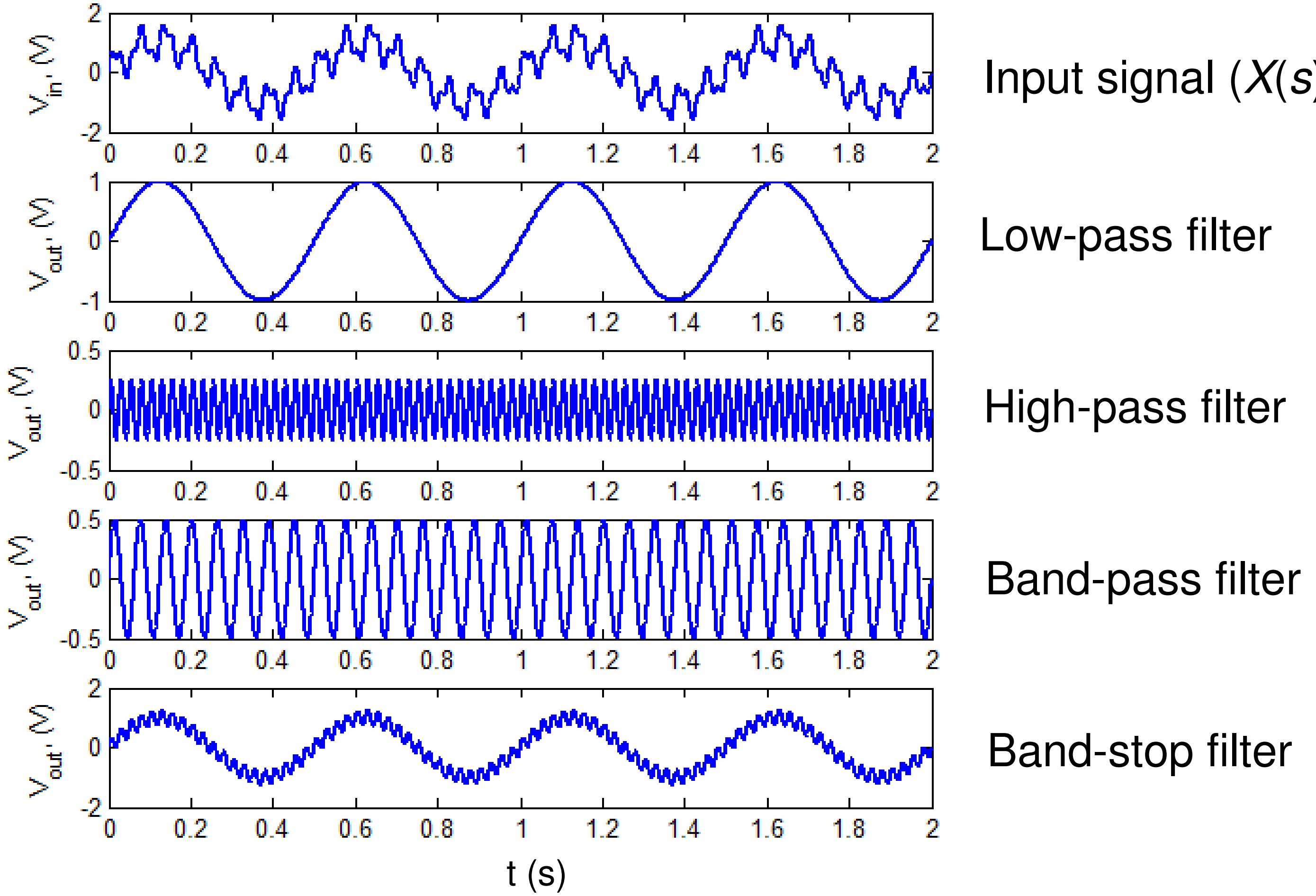
# Introduction

- **Definition** of a filter: A device (i.e. quadripole) that selects a interval of frequencies from an input signal, whose amplitudes and phase can be modified.
- **Analogical** filters (signal is no discrete)
  - **Passive**: made only with R,L and C
  - **Active**: have also operational amplifiers, transistors,…
- **Digital** filters (signal is discrete)

# Introduction

Classification according to the selected range of frequencies

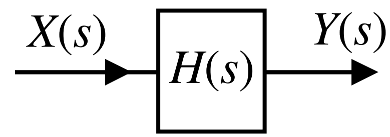


# Transfer Function *H*(*s*)

- Definition (Laplace Transform domain ($\mathcal{L}$))

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\mathcal{L}[\text{output signal}]}{\mathcal{L}[\text{input signal}]},$$

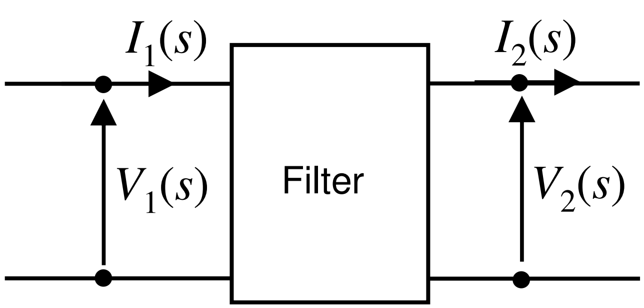



with initial conditions equal zero ($v_C(0)=i_L(0)=0$).

*H*(*s*) is independent of the applied signal.

Note that $H(s)=Y(s)$ when $x(t)=\delta(t)$, ($h(t)=\mathcal{L}^{-1}[H]$ is the *impulse response*).

- For example

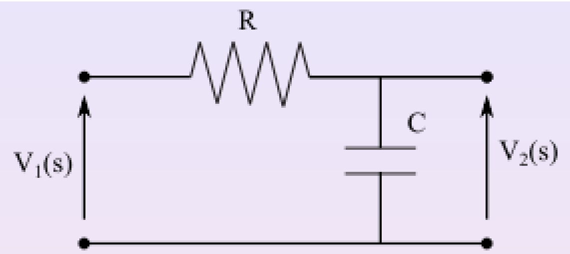



$$G_V(s) = \frac{V_2(s)}{V_1(s)}, \quad Z_T(s) = \frac{V_2(s)}{I_1}$$

$$G_I(s) = \frac{I_2(s)}{I_1(s)}, \quad Y_T(s) = \frac{I_2(s)}{V_1(s)}$$

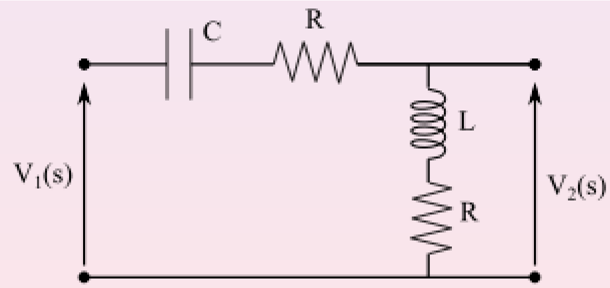
# Transfer Function

- Example 1




$$H_1(s) = \frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{RC} \cdot \frac{1}{s + \frac{1}{RC}}$$

- Example 2




$$H_2(s) = \frac{V_2(s)}{V_1(s)} = \frac{R + sL}{sL + 2R + \frac{1}{sC}} = \frac{s^2 + \frac{R}{L}s}{s^2 + \frac{2R}{L}s + \frac{1}{LC}}$$

What kind of filter are they?

# Poles and zeros of *H*(*s*)

- *H*(*s*) can be written as a fraction of two polynomials with real-valued and positive coefficients:

$$H(s) = K\frac{s^M + a_{M-1}s^{M-1} + \ldots + a_1}{s^N + b_{N-1}s^{N-1} + \ldots + b_1} = K\frac{\prod_M (s - c_m)}{\prod_N (s - p_n)}$$

where $c_m$ and $p_m$ are the **zeros** and the **poles** of the transfer function:

$$\lim_{s \to c_m}\{H(s)\} \to 0$$

$$\lim_{s \to p_m}\{H(s)\} \to \infty$$

zeros and the poles can also be at $s \to \infty$

# Pole-zero plot

- The poles and zeros are usually represented in a complex plane called the **pole-zero plot** to help to convey certain properties of the circuit
- The poles and ceros are either real or complex conjugated
- Poles are represented with “x”
- Zeros are represented with “o”

# Frequency response of the Transfer Function: $H(\omega)$

- Frequency response using sinusoidal signals, then $s$=j$\omega$ :

$$H(\omega) = K\frac{\prod_M (\mathrm{j}\omega - c_m)}{\prod_N (\mathrm{j}\omega - p_n)} = |H(\omega)|\exp(\mathrm{j}\phi(\omega))$$

where

$$|H(\omega)| = K\frac{\prod_M |\mathrm{j}\omega - c_m|}{\prod_N |\mathrm{j}\omega - p_n|},$$ is the amplitude response,

$$\phi(\omega) = \arctan\frac{\mathrm{Im}[H(\omega)]}{\mathrm{Re}[H(\omega)]},$$ is the phase response.

- $|H(\omega)|$ becomes high for $\omega$ close to $p_n$
- $|H(\omega)|$ becomes low for $\omega$ close to $c_m$

# Pole-zero plot and $|H(\omega)|$

- Example 3

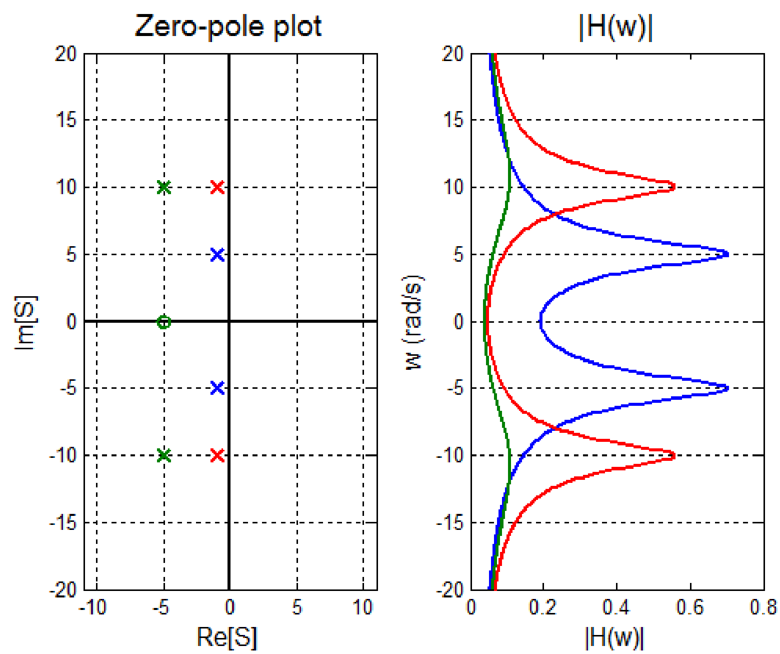
$$H(s) \propto \frac{s - c_1}{(s - p_1)(s - p_2)},$$

Poles at $s = p_1, p_2$
Zero at $s = c_1$

Poles at $s = -1 \pm j5$
Zero at $s = -5$

Poles at $s = -1 \pm j10$
Zero at $s = -5$

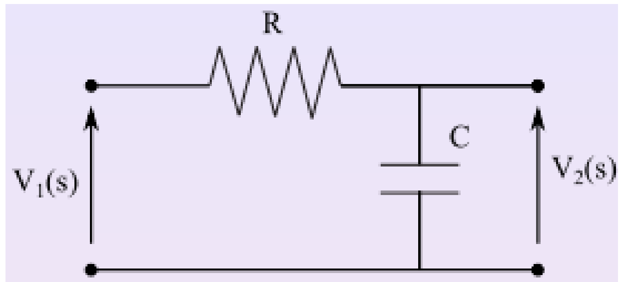
Poles at $s = -5 \pm j10$
Zero at $s = -5$




The closer the pole to the imaginary axis, the more pronounced is the maximum

# Pole-zero plot and $|H(\omega)|$

- Example 1



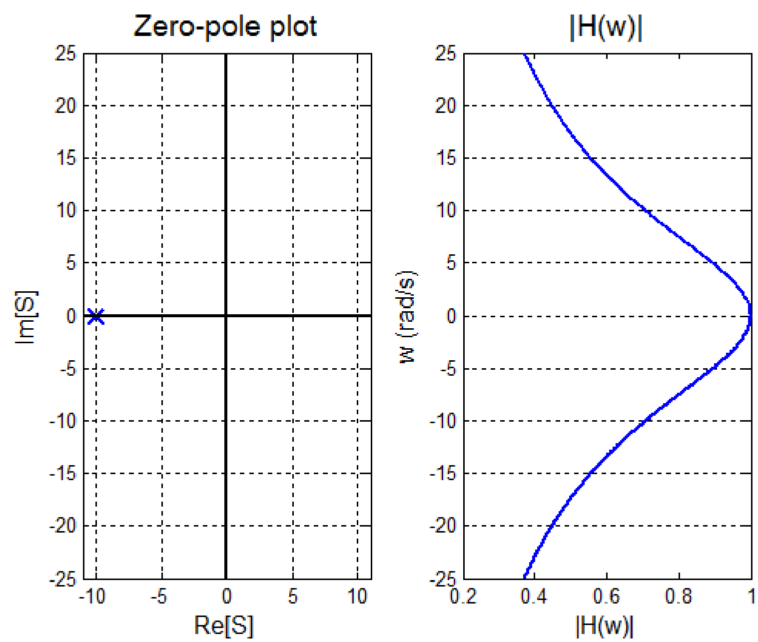

Data: R=1 kΩ, C=100 μF

$$H(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} = \frac{10}{s+10},$$

Pole at $s$= -10
Zero at $s \to \infty$

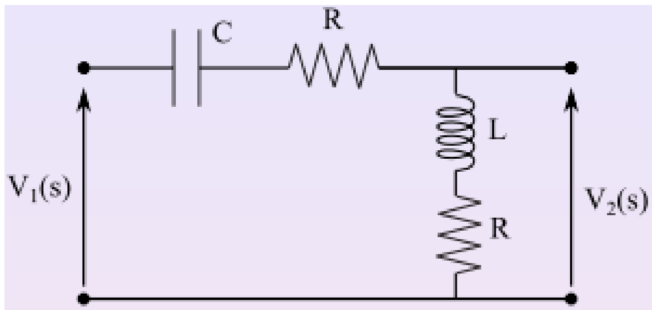
$$H(\omega) = \frac{10}{j\omega + 10} = \frac{10}{\sqrt{100+\omega^2}} \exp(j\phi)$$

$$\phi = \arctan \frac{-\omega}{10}$$



# Pole-zero plot and $|H(\omega)|$

- Example 2

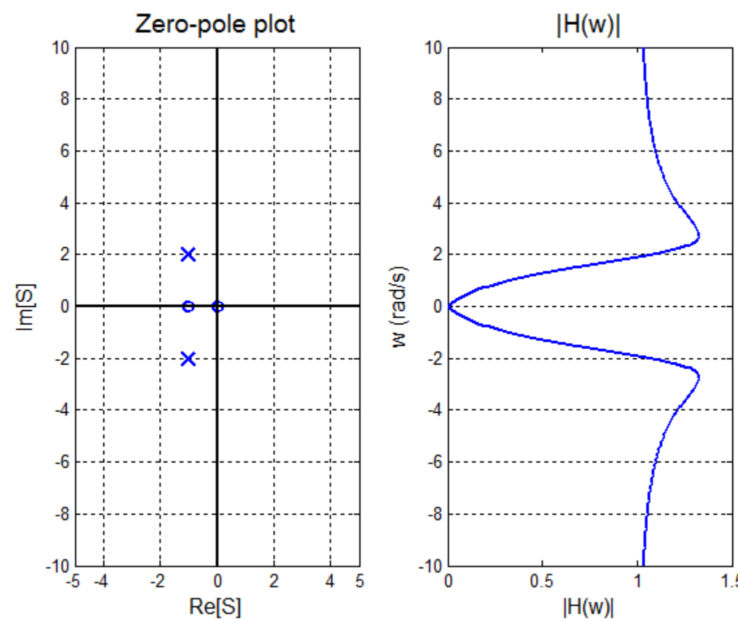



Data: R=1 Ω, C=1/5 F, L=1 H

$$H(s)=\frac{s^2+\frac{R}{L}s}{s^2+\frac{2R}{L}s+\frac{1}{LC}}=\frac{s(s+1)}{(s+1-\mathrm{j}2)(s+1+\mathrm{j}2)},$$

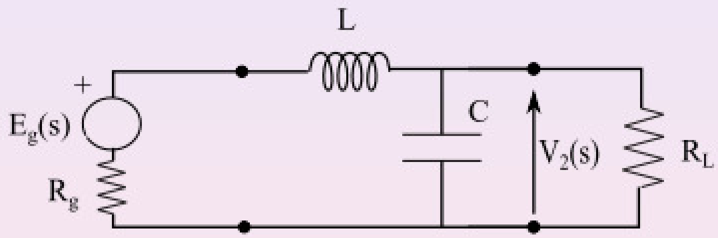
Poles at $s = -1\pm \mathrm{j}2$
Zeros at $s = 0,-1$

$$H(\omega)=\frac{-\omega^2+\mathrm{j}\omega}{5-\omega^2+\mathrm{j}2\omega},\quad |H(\omega)|=\frac{\omega\sqrt{1+\omega^2}}{\sqrt{(5-\omega^2)^2+4\omega^2}}$$



# $H(s)$ between $(E_g, R_g)$ and $R_L$


$$H(s) = \frac{V_2(s)}{E_g(s)}$$



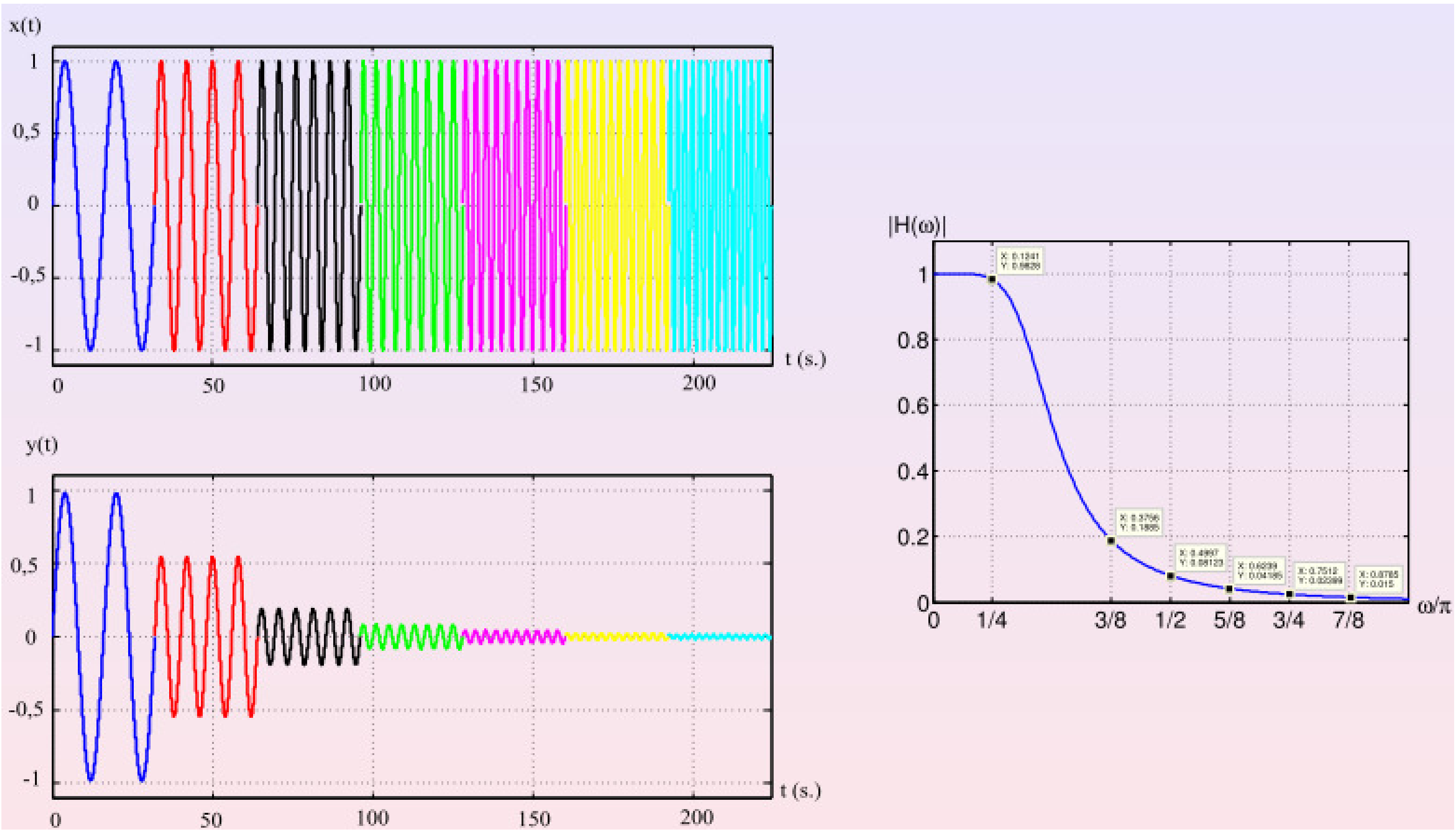

$$\frac{V_2(s) - E_g(s)}{R_g + sL} + sC\,V_2(s) + \frac{V_2(s)}{R_L} = 0$$

$$H(s) = \frac{V_2(s)}{E_g(s)} = \frac{R_L}{R_g + R_L} \cdot \frac{\frac{R_g+R_L}{LCR_L}}{s^2 + \left(\frac{1}{CR_L} + \frac{R_g}{L}\right) s + \frac{R_g+R_L}{LCR_L}}$$

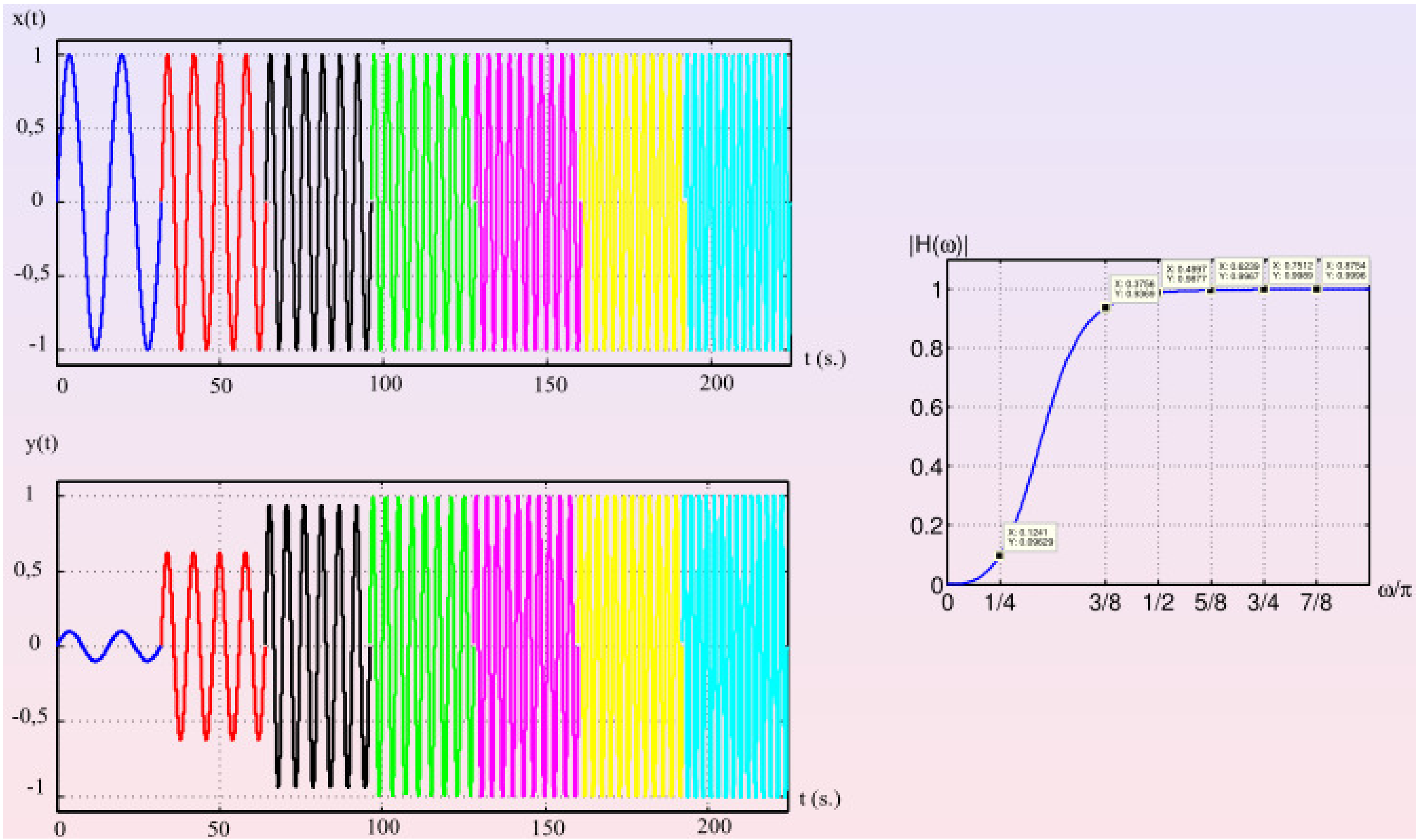
# RLC passive filters

- **Low-pass** filter
  - 1st order
  - 2nd order
- **High-pass** filter
  - 1rt order
  - 2nd order
- **Band-pass** filter (2nd order)
- **Band-stop** filter (2nd order)

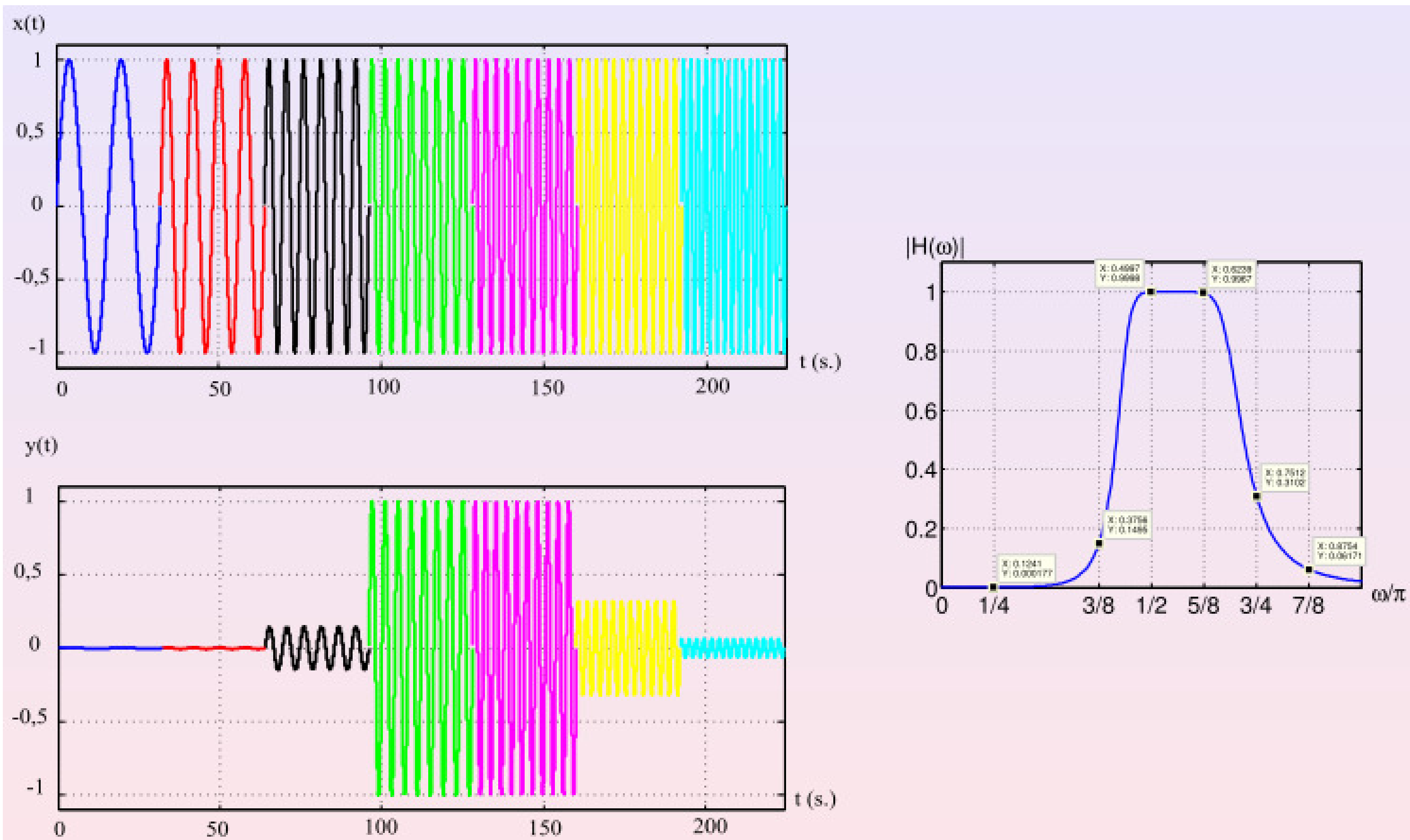
# Example: Low-pass filer



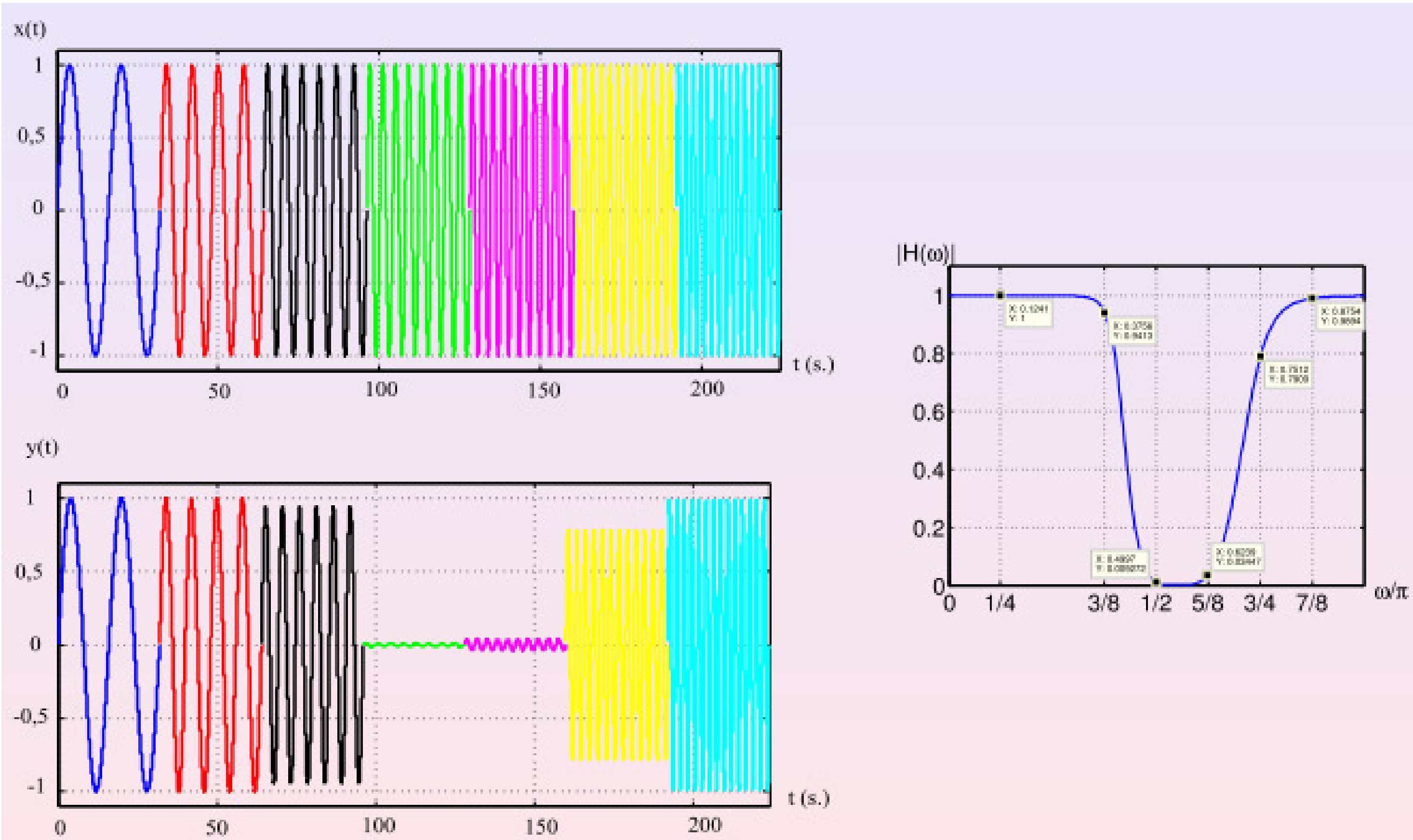
# Example: High-pass filer



# Example: Band-pass filer

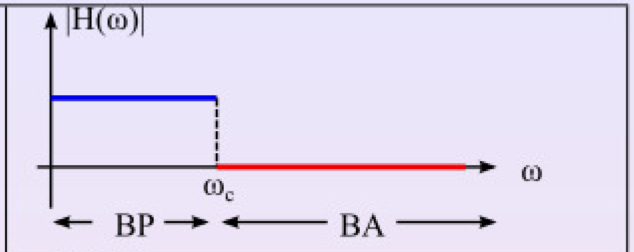


# Example: Band-stop filter

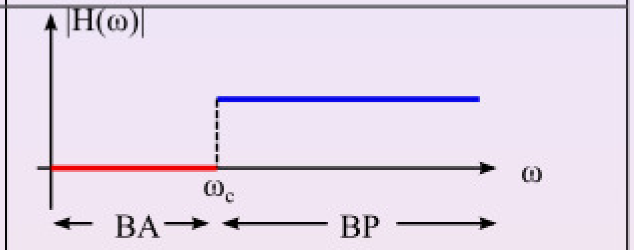


# Ideal amplitude response

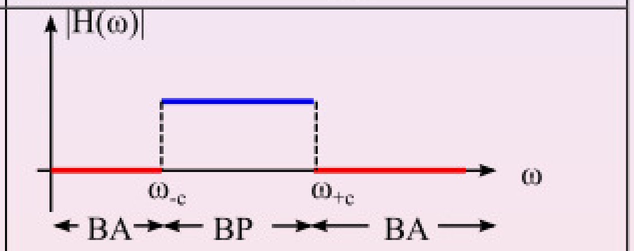
Low-pass filter



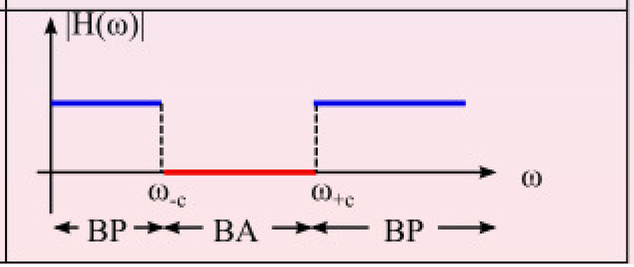

High-pass filter




Band-pass filter




Band-stop filter

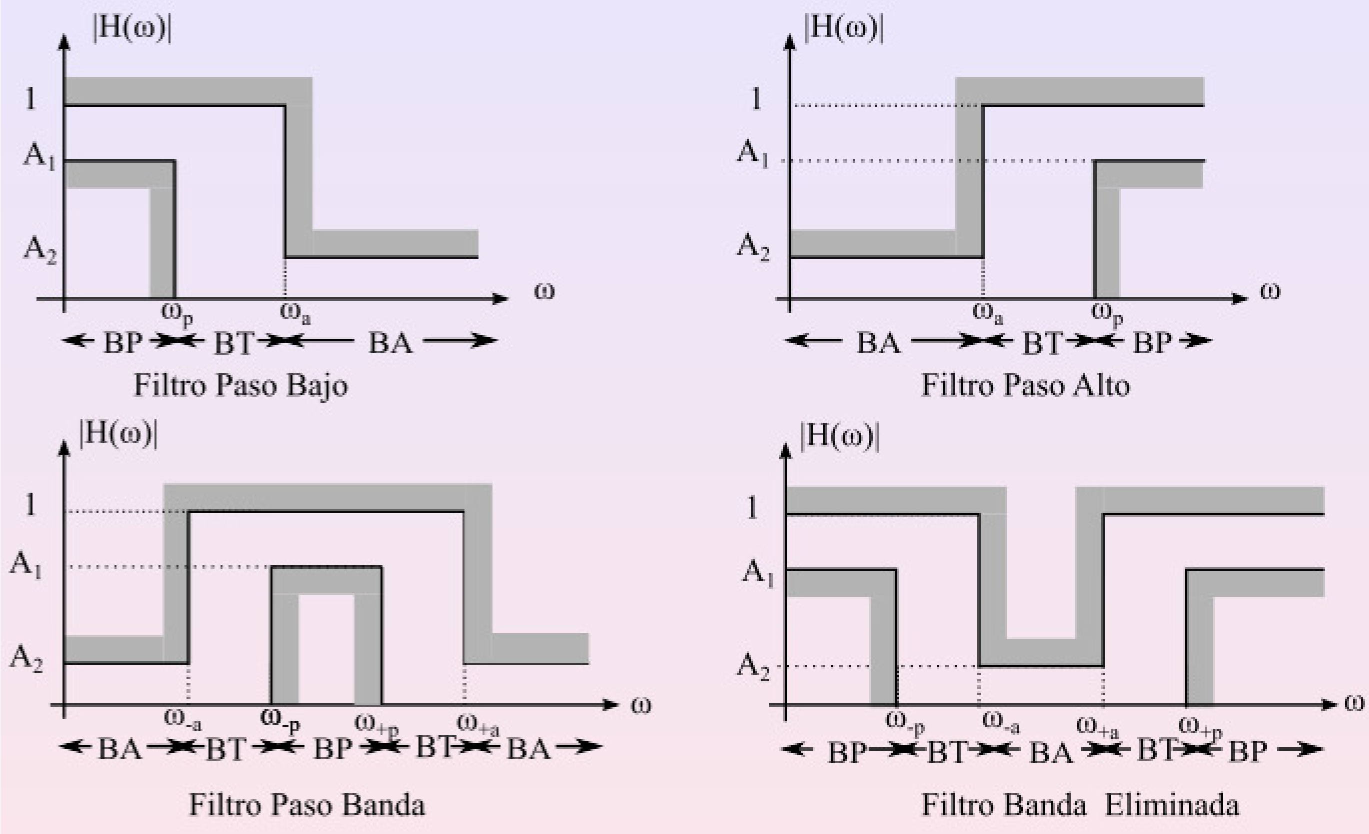



PB: Pass band, AB: Attenuation band, $\omega_C$: Cut frequency

# Impulse response of ideal filters

- Ideal filters can not be implemented in practice
- The real filer have to be approached to the ideal filter response using different methods
- The approached filters have not a constant response in the PB and are not entirely zero in the AB
- There exist a transition band between PB and AB
- The **order of the filter** coincides with the number of poles of H(s)
- The higher the order of the filter the close is his behavior as an ideal filter
- To design filters, layout templates with the tolerance margins are used

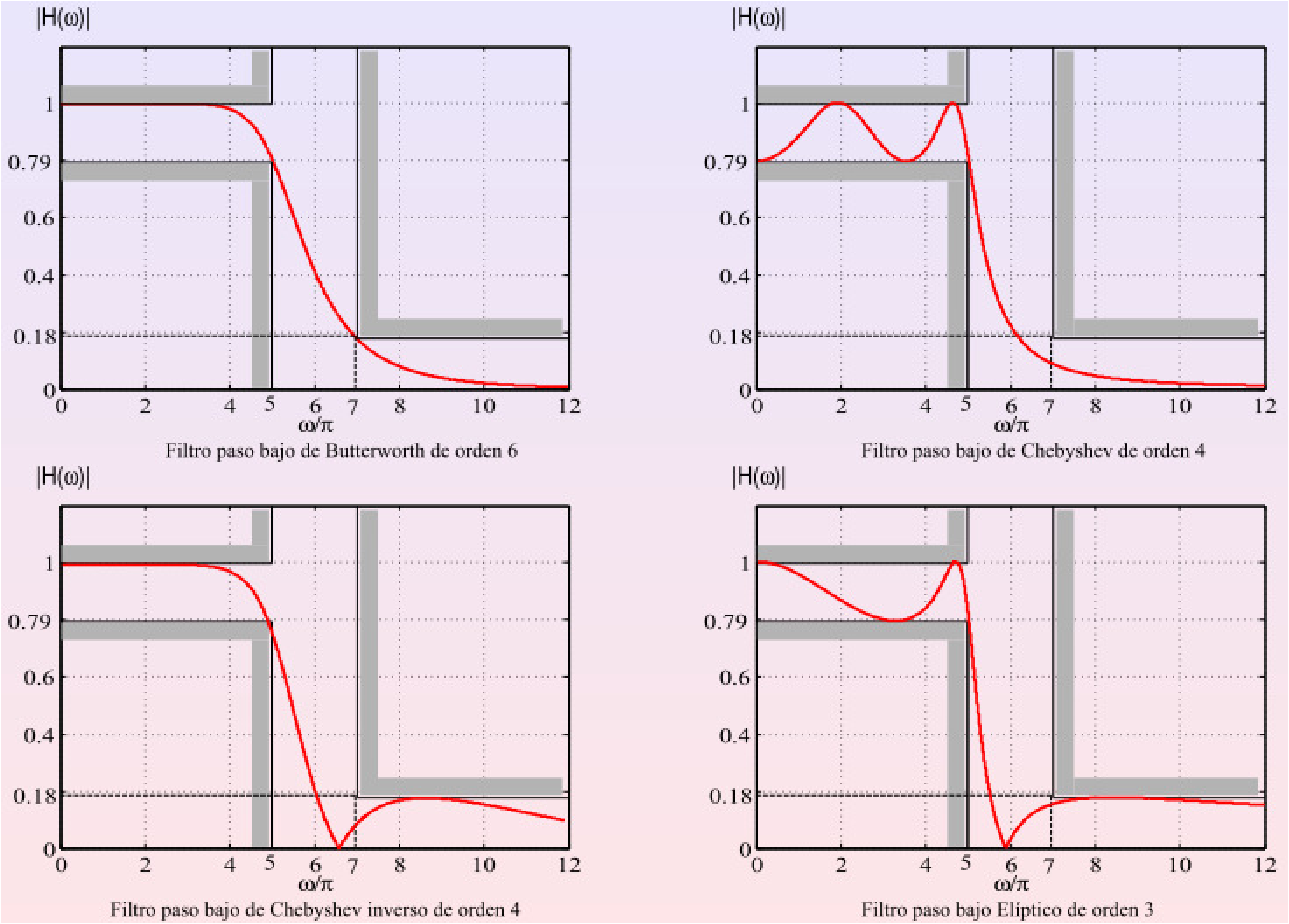
# Layout templates for filter design




$\omega_p$ : Pulsación de corte de la banda de paso ; $\omega_a$ : Pulsación de corte de la banda atenuada

$A_1$ : Amplitud mínima en la banda de paso ; $A_2$ : Amplitud máxima en la banda atenuada

# Examples of real filters



# 1st order low-pass filter

- Transfer function

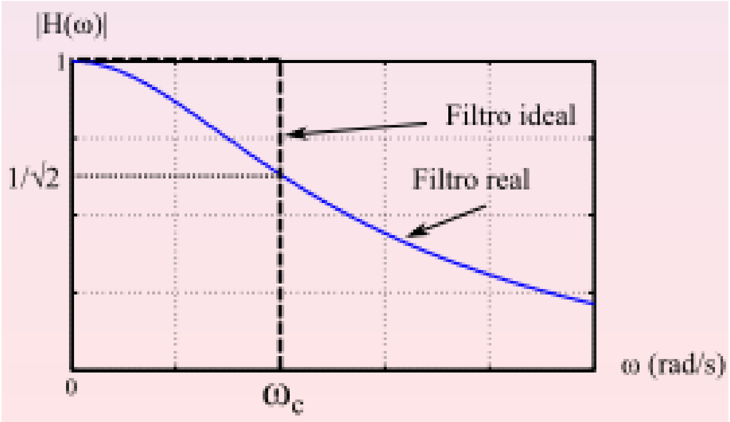
$$H(s) = k\frac{\omega_c}{s+\omega_c}$$

where $\omega_c$ is the **cut frequency,** at this frequency the **attenuation** of the signal is **3dB:**

$$-20\log_{10}\frac{|H(\omega)|_{\omega=\omega_c}}{|H(\omega)|_{max}} = 3 \Rightarrow \frac{|H(\omega)|_{\omega=\omega_c}}{|H(\omega)|_{max}} = \frac{1}{\sqrt{2}}$$

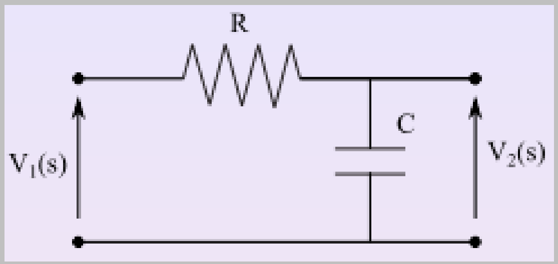
$$\Leftrightarrow |H(\omega)|_{\omega=\omega_c} = |H(\omega)|_{max}10^{\frac{-3}{20}}$$

$$|H(\omega)| = k\frac{\omega_c}{\sqrt{\omega^2+\omega_c^2}}$$



# Examples 1st order low-pass filter

- Low-pass RC:



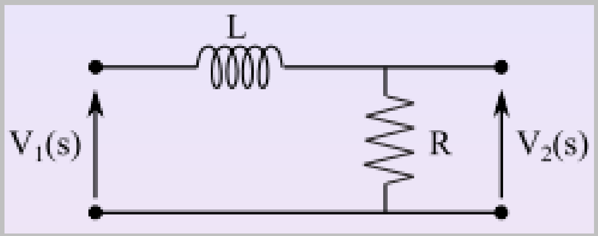

$$Z_C(\omega \to \infty) \to 0 \Rightarrow V_2 \to 0$$

$$Z_C(\omega = 0) \to \infty \Rightarrow V_2 \to V_1$$

$$H(\omega) = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}},$$

$$\omega_c = \frac{1}{RC}$$

- Low-pass RL




$$Z_L(\omega \to \infty) \to \infty \Rightarrow V_2 \to 0$$

$$Z_L(\omega = 0) = 0 \Rightarrow V_1 = V_2$$

$$H(\omega) = \frac{\frac{R}{L}}{j\omega + \frac{R}{L}},$$

$$\omega_c = \frac{R}{L}$$

# 1st order high-pass filter

- Transfer function

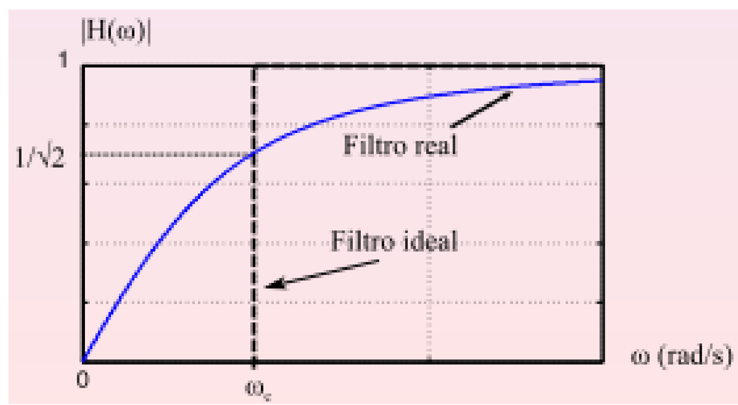
$$H(s) = k\frac{s}{s+\omega_c}$$

where $\omega_c$ is the **cut frequency,** at this frequency the **attenuation** of the signal is **3dB:**

$$-20\log_{10}\frac{\left|H(\omega)\right|_{\omega=\omega_c}}{\left|H(\omega)\right|_{\max}} = 3 \Rightarrow \frac{\left|H(\omega)\right|_{\omega=\omega_c}}{\left|H(\omega)\right|_{\max}} = \frac{1}{\sqrt{2}}$$

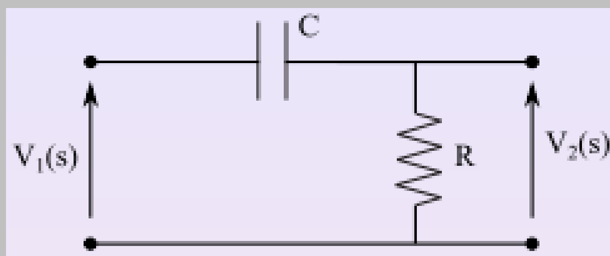
$$\left|H(\omega)\right|_{\omega=\omega_c} = \left|H(\omega)\right|_{\max} 10^{\frac{-3}{20}}$$

$$\left|H(\omega)\right| = k\frac{\omega}{\sqrt{\omega^2+\omega_c^2}}$$



# Examples 1st order high-pass filter

- High-pass RC:

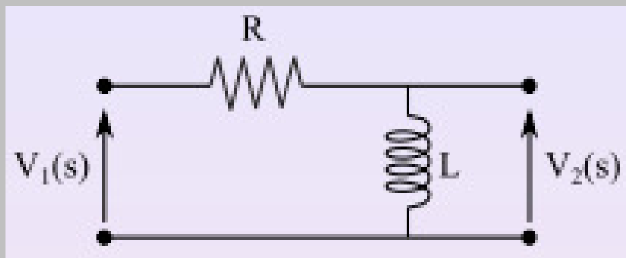



$$Z_C(\omega = 0) \to \infty \Rightarrow V_2 \to 0$$
$$Z_C(\omega \to \infty) = 0 \Rightarrow V_2 = V_1$$

$$H(\omega) = \frac{j\omega}{j\omega + \frac{1}{RC}},$$

$$\omega_c = \frac{1}{RC}$$

- High-pass RL




$$Z_L(\omega = 0) = 0 \Rightarrow V_2 = 0$$
$$Z_L(\omega \to \infty) \to \infty \Rightarrow V_2 \to V_1$$

$$H(\omega) = \frac{j\omega}{j\omega + \frac{R}{L}},$$

$$\omega_c = \frac{R}{L}$$

# 2st order passive filters

- Transfer function:

$$H(s) = k\,\frac{N_2(s)}{s^2 + a_1 s + a_2} = k\,\frac{N_2(s)}{s^2 + 2\xi\omega_0 s + \omega_0^2} = k\,\frac{N_2(s)}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

- $N_2(s)$: Polynomial of order $\leq 2$
- $\xi$: **Damping coefficient**
- $\omega_0$: **Characteristic frequency** of the filter
- $Q=1/(2\xi)$: **Quality factor** of the circuit
- *For band-pass and band stop filters:*

  $\boldsymbol{B} = \omega_0/Q$: **Band width** of the circuit

  $$k = \left|H(\omega)\right|_{\max} (= H_0)$$

# 2nd order passive filters

- With 2nd order filter the four type of filters can be obtained (with 1st order only two)
- The poles of *H*(*s*) are:

$$s_{p1,2} = -\omega_0\left(\xi \pm \mathrm{j}\sqrt{1-\xi^2}\right) = -\frac{\omega_0}{2Q}\left(1 \pm \mathrm{j}\sqrt{4Q^2-1}\right)$$

- Depending on the value of ξ (or Q) we distinguish the following two poles:
  - Real and different (ξ>1 or Q<1/2)
  - Real and equal (ξ=1 or Q=1/2)
  - Complex conjugated (ξ<1 or Q>1/2)
- Depending on $N_2(s)$ we have the following special cases:

# 2nd order filters

- Low-pass filter

$$H(s) = k \cdot \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

- High-pass filter

$$H(s) = k \cdot \frac{s^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

- Band-pass filter

$$H(s) = k \cdot \frac{\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

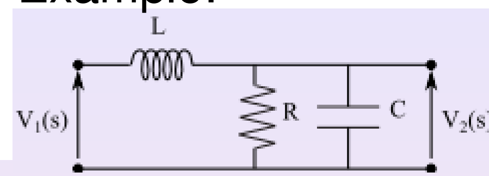
- Band-stop filter

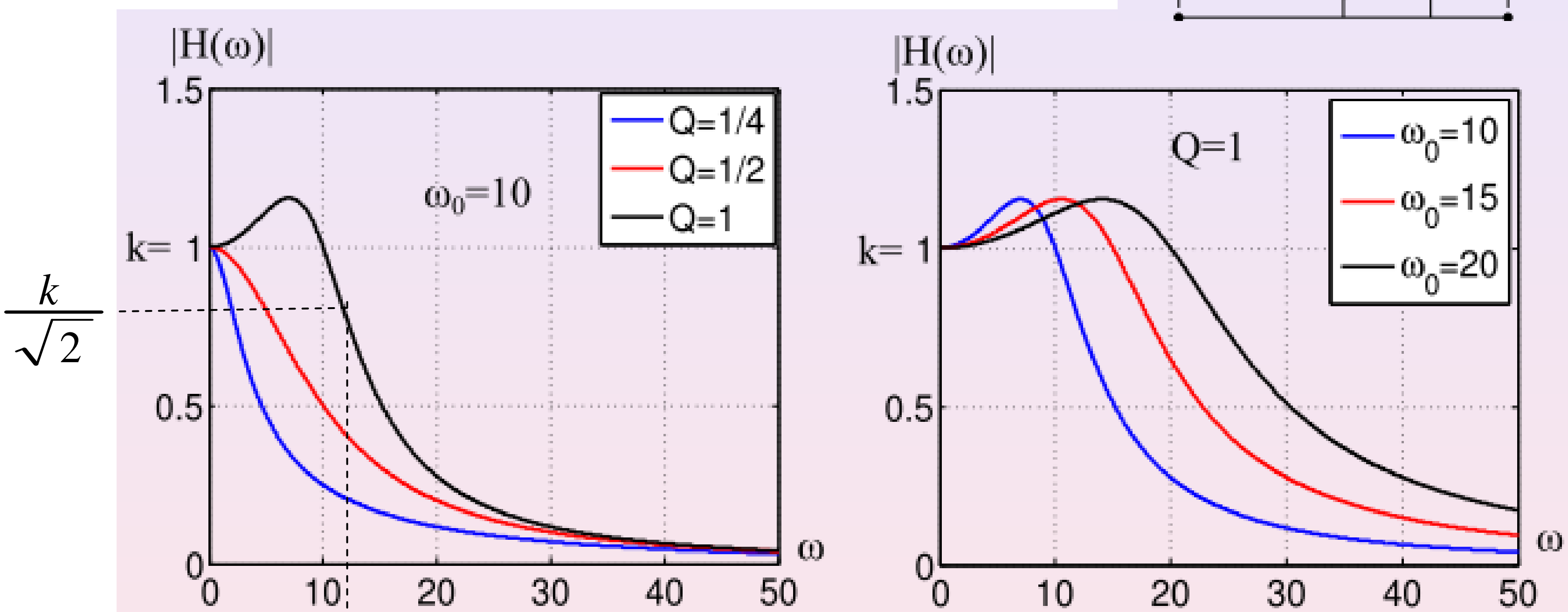
$$H(s) = k \cdot \frac{s^2 + \omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

# 2nd order low-pass filter

$$H(s) = k\frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

Example:



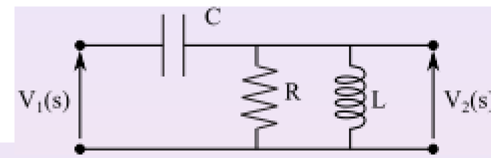




***Cut frequency:*** $\omega_C$

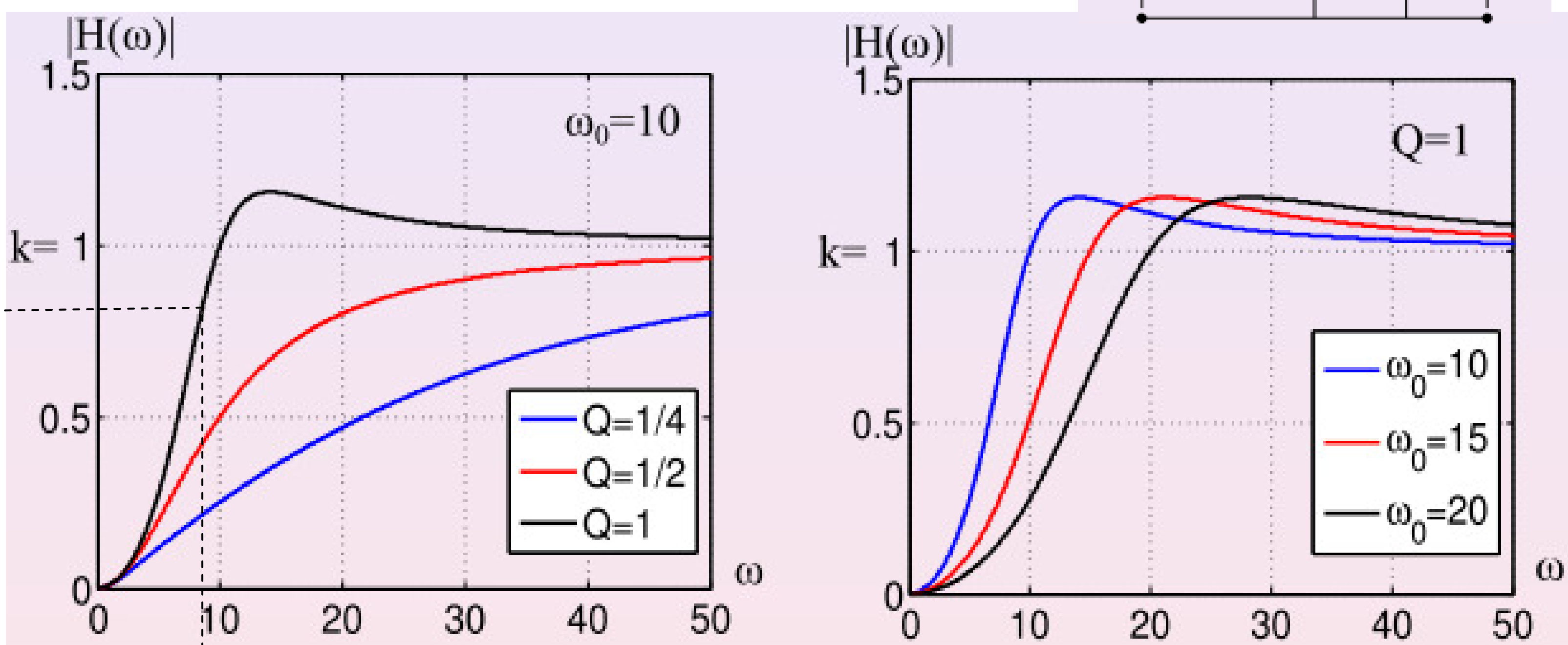
# 2nd order high-pass filter

$$H(s) = k \frac{s^2}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

Example:




$$\frac{k}{\sqrt{2}}$$

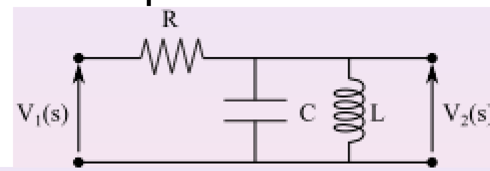


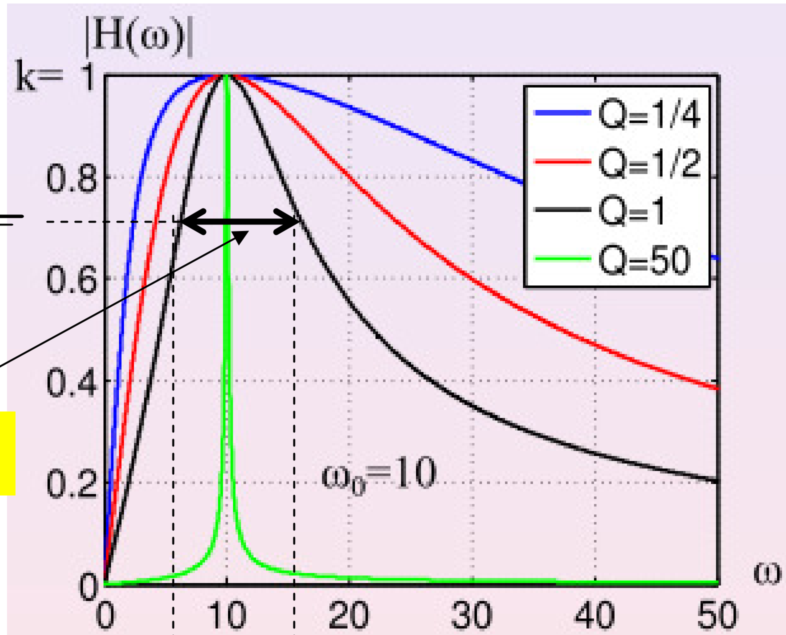

***Cut frequency:*** $\omega_C$

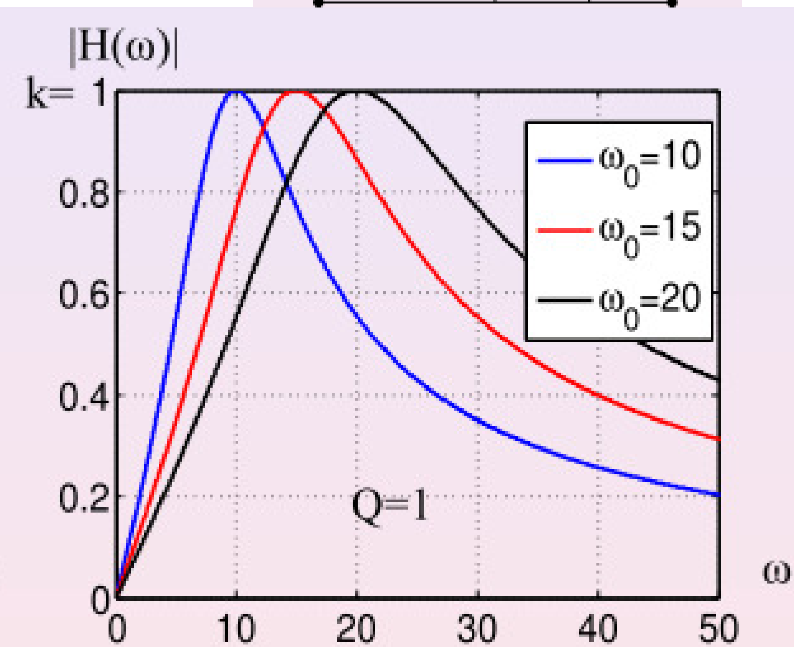
# 2nd order band-pass filter

$$H(s) = k\frac{\frac{\omega_0}{Q}s}{s^2+\frac{\omega_0}{Q}s+\omega_0^2} = k\frac{Bs}{s^2+Bs+\omega_0^2}$$

Example:




$\frac{k}{\sqrt{2}}$

**B=$\omega_0$/Q**

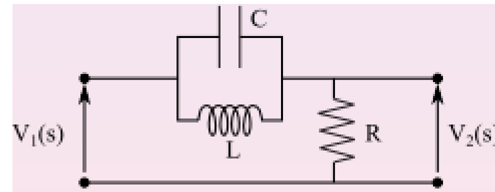
***Cut frequencies:*** $\omega_{C,1}$ $\omega_{C,2}$

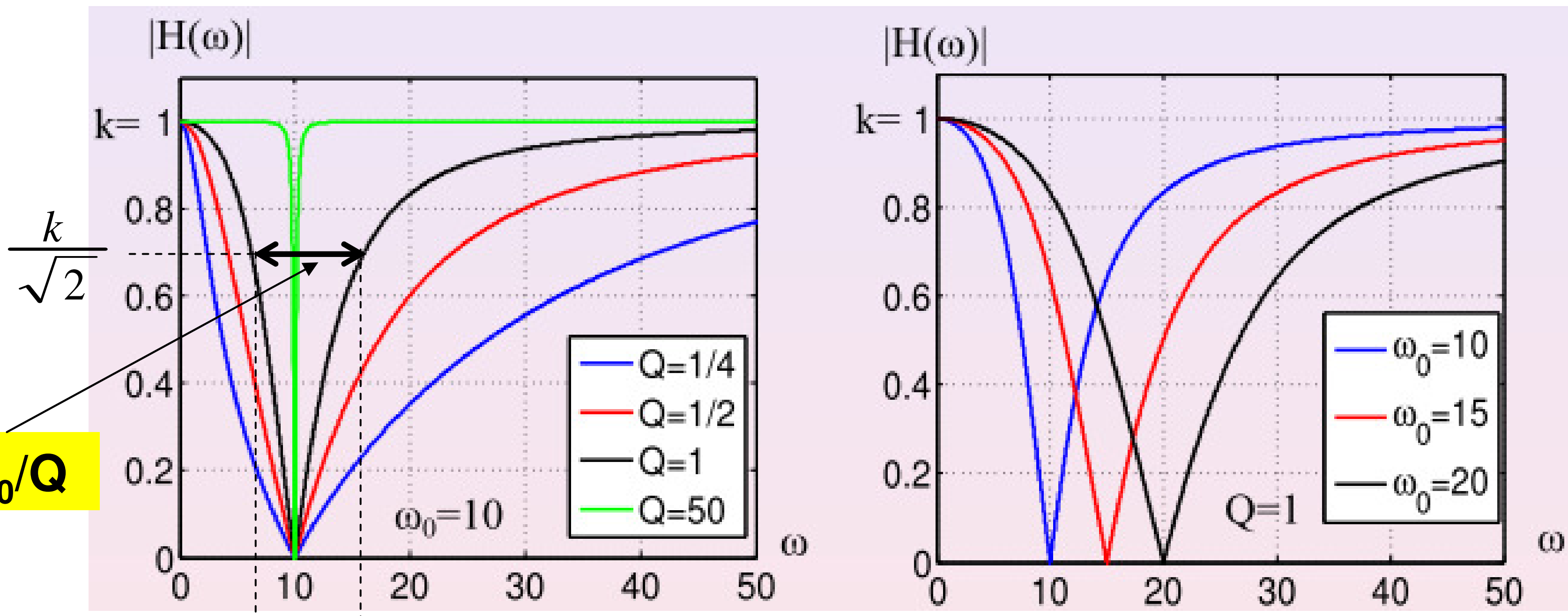
**B=$\omega_{C,2} - \omega_{C,1}$**

# 2nd order Band-stop filter

$$H(s) = k\frac{s^2+\omega_0^2}{s^2+\frac{\omega_0}{Q}s+\omega_0^2}$$

Example:



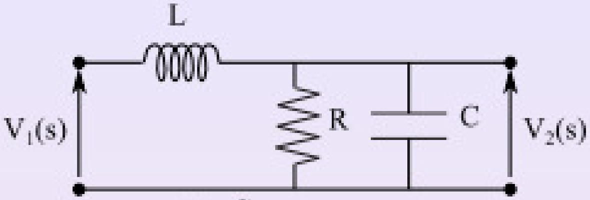




**B=$\omega_0$/Q**

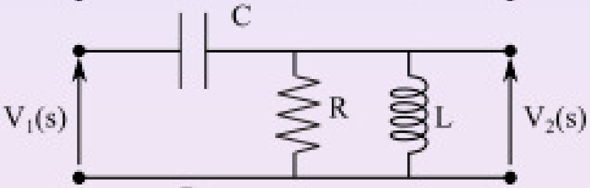
***Cut frequencies: $\omega_{C,1}$*** ***$\omega_{C,2}$***

***B=$\omega_{C,2} - \omega_{C,1}$***

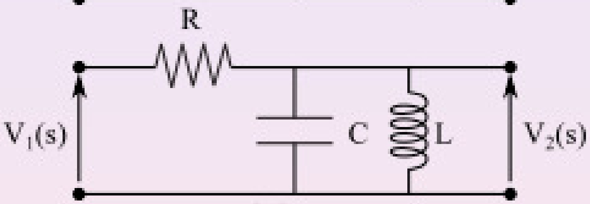
# Examples 2nd order filters



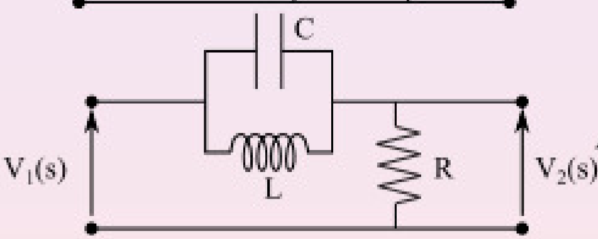

$$H(s) = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$ F.P. Bajo




$$H(s) = \frac{s^2}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$ F.P. Alto




$$H(s) = \frac{\frac{1}{RC}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$ F.P. Banda




$$H(s) = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$ F.Banda Eliminada

Siendo: $\omega_0 = \frac{1}{\sqrt{LC}}$ ; $Q = R\omega_0 C = \frac{R}{\omega_0 L} = R\sqrt{\frac{C}{L}}$