Module 2. Sinusoidal Steady State Circuit analysis

Circuit Theory (**350004**) , 2018-19

Outline

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- Why complex numbers?
- Phasors and impedances
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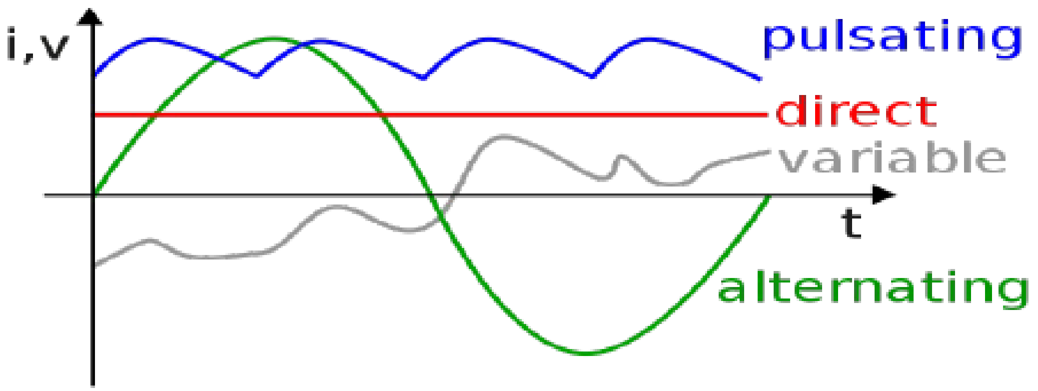
The sinusoidal signal

- Most electrical sources are **Alternating Current** (**AC**) which periodically reverses direction, and is the form in which electrical power is delivered.
- The usual waveform of AC is the sine wave, which is called **Sinusoidal Steady State** (**SSS**) regimen.



SSS signals

- Sinusoidal sources are particularly important because:
  - Generation, transmission, consumption of electric energy occur under sinusoidal conditions.
  - It can be used to predict the behaviors of circuits with non-sinusoidal sources (applying principle of superposition)
- We will work with complex numbers (i.e. phasors and impedances). https://en.wikipedia.org/wiki/Complex_number

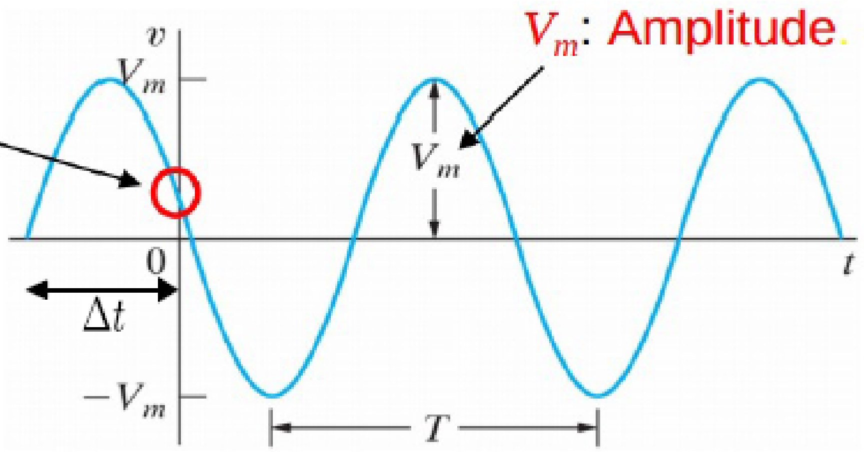
SSS signals

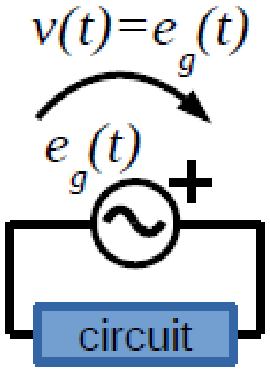
- A source that provides a sinusoidal signal is then reproduced (after the transient behavior) by all the currents and voltages throughout the circuit with the same frequency but with their corresponding amplitudes and phases

$$v(t) = V_m \sin(\omega t + \phi), \quad \omega = \frac{2\pi}{T}$$

ϕ: Phase angle, determines the value at $t = 0$.

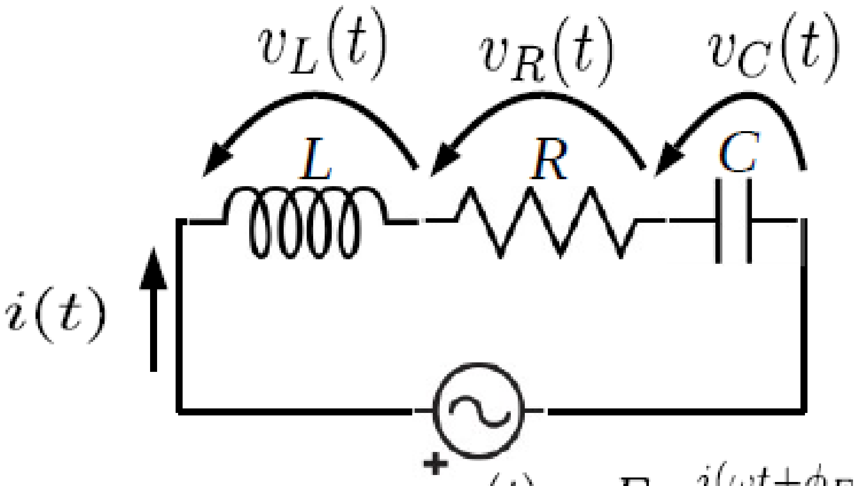
$$\phi = \frac{2\pi}{T}\Delta t$$





Why complex numbers?

Why using complex numbers?. Let's see it in the following example of a simple circuit with a SSS source (excited with a specific frequency "ω")



$$v_L(t) + v_R(t) + v_C(t) = e(t)$$

$$\Rightarrow L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}q(t) = e(t)$$

$$\Rightarrow \frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{de(t)}{dt}$$

$$e(t) = E_0 e^{j(\omega t + \phi_E)} = E_0 \cos(\omega t + \phi_E) + jE_0 \sin(\omega t + \phi_E)$$

To solve the circuit means to solve the 2nd order differential equation

Why complex numbers?

Since the source is: $e(t) = E_0 e^{j(\omega t + \phi_E)}$

the current will be : $\Rightarrow \quad i(t) = I_0 e^{j(\omega t + \phi_I)}$

where I_0 and ϕ_I are unknown

Introducing this "predicted form of solution" into the differential equation delivers:

$$\left(j\omega L + R + \frac{1}{j\omega C}\right) I_0 e^{j\phi_I} e^{j\omega t} = E_0 e^{j\phi_E} e^{j\omega t}$$

Are the so called **Impedances** and will be our new parameters for the electrical elements.

Are the so called **Phasors** and will be our new variables.

Is the temporal evolution or **carrier** and is **the same** for all the signals (currents or voltages) in the circuit (supposing all the sources are SSS with the same ω.)

Phasors and Impedances

- By defining the following complex numbers we will simplify our circuit's resolution
- **Phasors** of the
  - Current
  - Voltage

$$\mathbf{I} = I_0 e^{j\phi_I}$$

$$\mathbf{V} = V_0 e^{j\phi_V}$$

The phasors contain the information of both:

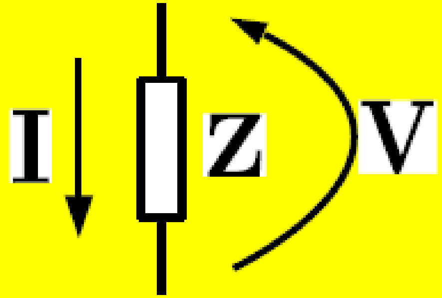
- **Amplitude**
- **Phase**

of the temporal expression

e.g. the electromotive force of a voltage source: $\mathbf{E} = E_0 e^{j\phi_E}$

- **Impedance**

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$



(Ohm's Law in the “phasor's domain”)

Phasors and Impedances

- Introducing the phasors and skipping the carrier in the previous equation delivers

$$\left(j\omega L + R + \frac{1}{j\omega C}\right)\mathbf{I} = \mathbf{E}$$

which is an algebraic equation, much easier to solve !!

where

$$j\omega L\mathbf{I} = \mathbf{V_L}$$
$$R\mathbf{I} = \mathbf{V_R}$$
$$\frac{1}{j\omega C}\mathbf{I} = \mathbf{V_C}$$

are the respective phasors of the voltages at each element

Impedances of L and C

- The impedances of L, C and R are then

$$\mathbf{Z}_L = \frac{\mathbf{V}_L}{\mathbf{I}_L} = j\omega L$$

$$\mathbf{Z}_C = \frac{\mathbf{V}_C}{\mathbf{I}_C} = \frac{1}{j\omega C}$$

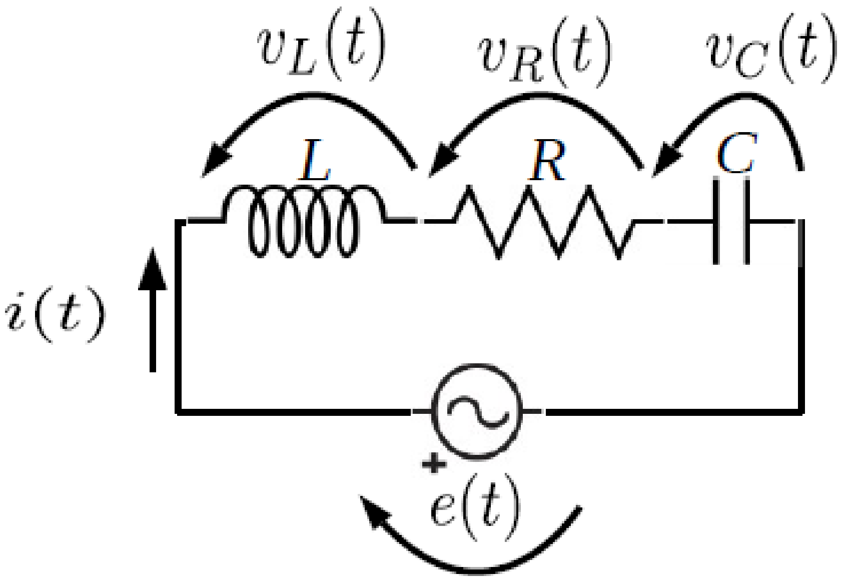
$$\mathbf{Z}_R = \frac{\mathbf{V}_R}{\mathbf{I}_R} = R$$

In **our example** of circuit with L, R and C connected in series, it can be seen that the phasor $\mathbf{E}$ is the sum of the voltage phasor's at each element: $\mathbf{V}_L + \mathbf{V}_R + \mathbf{V}_C = \mathbf{E}$.
And the current trough them is the same: $\mathbf{I} = \mathbf{I}_L = \mathbf{I}_R = \mathbf{I}_C$.

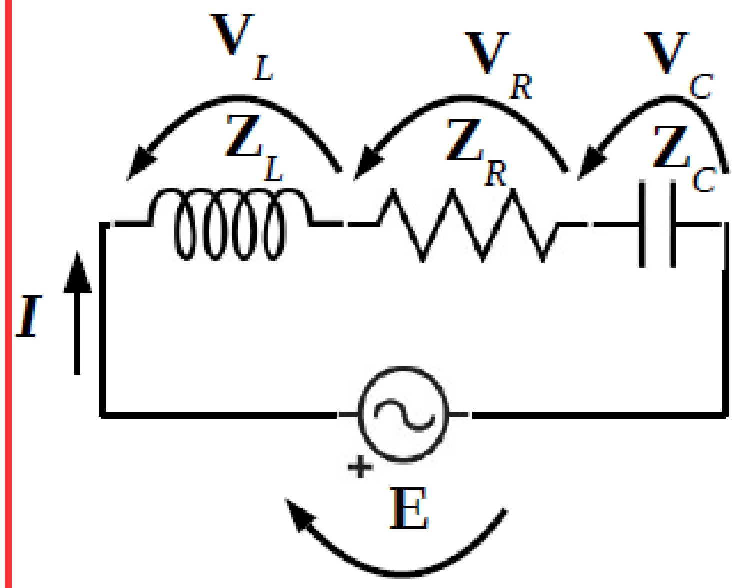
- Each passive electrical element can be characterized with a impedance obtained by combining (i.e. associating) these impedance's

The "phasor domain"

- Using the defined impedances and phasors, the resolution of the previous circuit simplifies to:



Phasor domain



$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC} i(t) = \frac{de(t)}{dt}$$

$$(\mathbf{Z}_L + \mathbf{Z}_R + \mathbf{Z}_C)\,\mathbf{I} = \mathbf{E}$$

The “phasor domain”

- The circuit's excited with SSS sources are then solved in the “phasor domain” (i.e. using impedances and phasors).
  - Coils and capacitors are characterized with their impedances (Resistors remain as they are)
  - Currents and voltages are “transformed” into their respective phasors
- When obtained the unknown phasor, the temporal expression is obtained by multiplying it by the carrier, for our example:

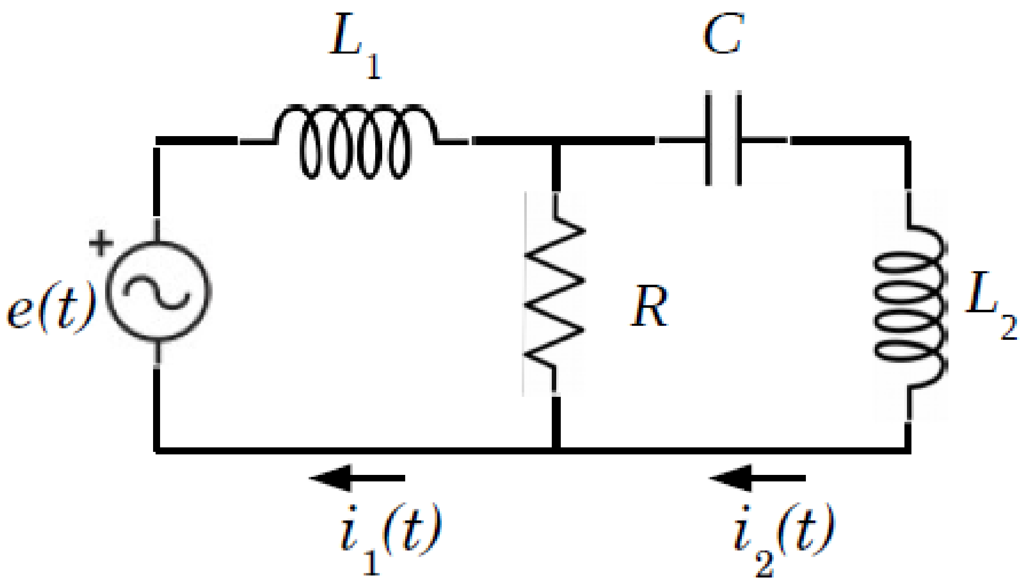
$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_L + \mathbf{Z}_R + \mathbf{Z}_C} = I_0 \mathrm{e}^{j\phi_I} \Rightarrow i(t) = \mathbf{I}\mathrm{e}^{j\omega t} = I_0 \mathrm{e}^{j(\omega t + \phi_I)}$$

In the phasor domain

- The **Kirchhoff's law's are** also **valid** for the phasors (thus, circuits are solved in the same way as in DC)
- Solving the circuit in the phasor domain is like solving it for a specific ***time instant*** for which the different “sinusoidal signals” have each their own phase and amplitude.
- The ***amplitude*** and the ***relative phase*** between the signals **does not change** with the time.

Example

- Calculate of the values of $i_1(t)$ and $i_2(t)$



Data:

$$e(t) = 3\sin(10^6 t + \frac{\pi}{2})\text{V}$$

$$R = 300\Omega$$

$$L_1 = 0.9\text{mH}$$

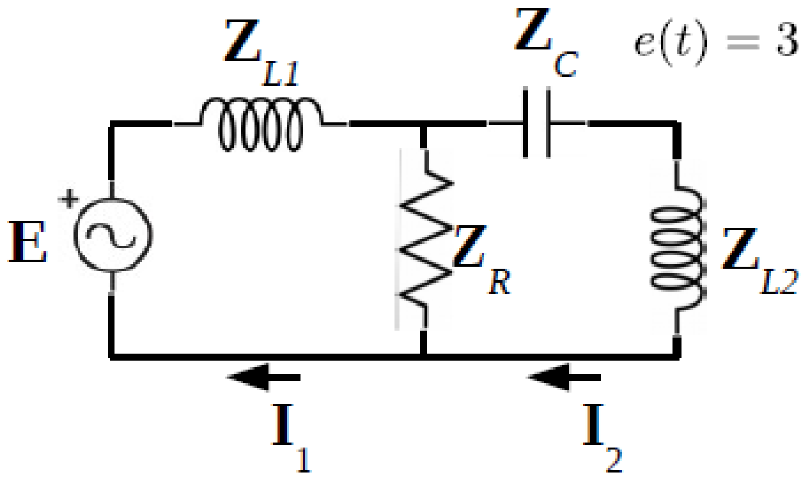
$$L_2 = 0.3\text{mH}$$

$$C = 20\text{nF}$$

We use capital letters for designing phasors from their corresponding time domain expressions where we use lower case letters. For example $\mathbf{I}$ versus $i_1(t)$.

Example

- 1) Translate into the phasor domain and solve



$e(t) = 3\sin(10^6 t + \frac{\pi}{2})\text{V} \Rightarrow \mathbf{E} = 3e^{j\frac{\pi}{2}} = j3\text{V}$

$R = 300\Omega \Rightarrow \mathbf{Z_R} = 300\Omega$

$L_1 = 0.9\text{mH} \Rightarrow \mathbf{Z}_{L1} = j \cdot 10^6 \cdot 0.9 \cdot 10^{-3} = j900\Omega$

$L_2 = 0.3\text{mH} \Rightarrow \mathbf{Z}_{L2} = j \cdot 10^6 \cdot 0.3 \cdot 10^{-3} = j300\Omega$

$C = 20\text{nF} \Rightarrow \mathbf{Z}_C = \frac{1}{j \cdot 10^6 \cdot 20 \cdot 10^{-9}} = -j50\Omega$

Applying Kirchhoff's Law's:

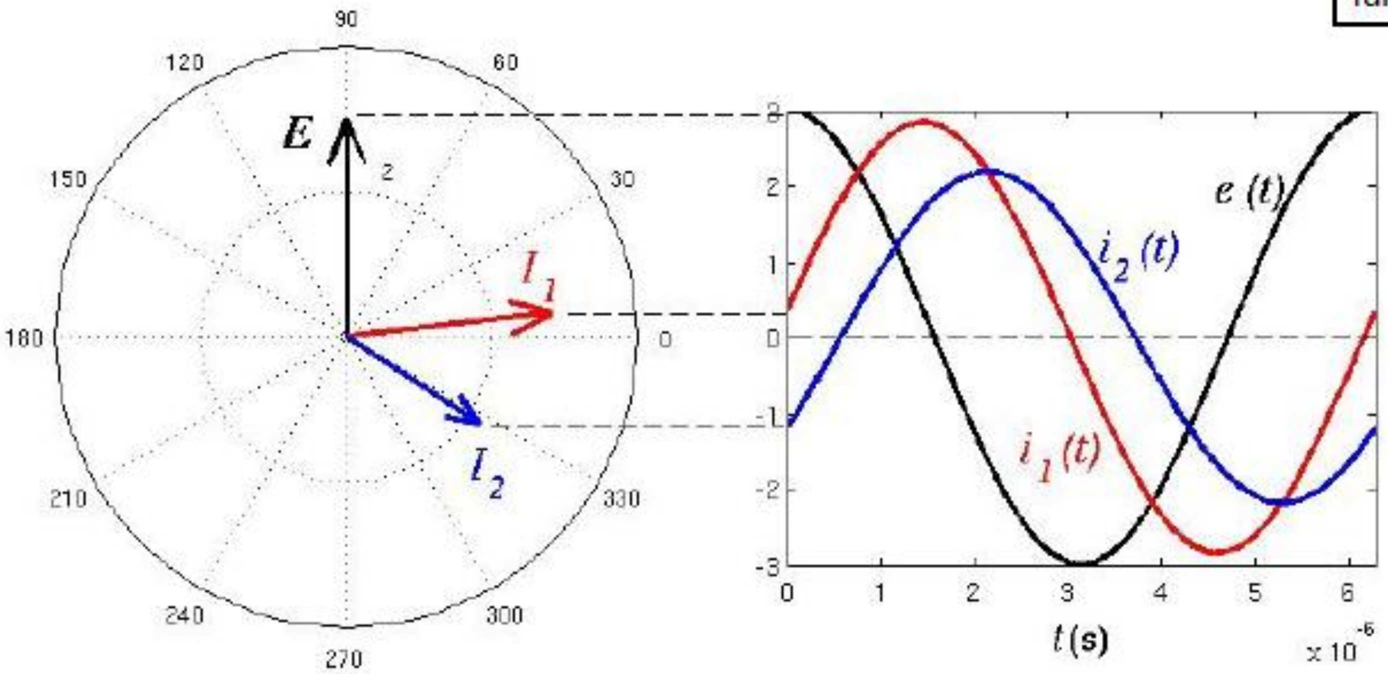
$$\begin{cases} \mathbf{E} = \mathbf{Z}_{L1}\mathbf{I}_1 + \mathbf{Z}_R(\mathbf{I}_1 - \mathbf{I}_2) \\ 0 = (\mathbf{Z}_C + \mathbf{Z}_{L2})\mathbf{I}_2 + \mathbf{Z}_R(\mathbf{I}_2 - \mathbf{I}_1) \end{cases} \Rightarrow \begin{cases} \mathbf{I}_1 = 2.82 + j0.33 = 2.84e^{j0.12}\text{mA} \\ \mathbf{I}_2 = 1.83 - j1.19 = 2.19e^{-j0.58}\text{mA} \end{cases}$$

Example

- 2) Move back to the "time domain" to obtain the temporal expressions of the currents

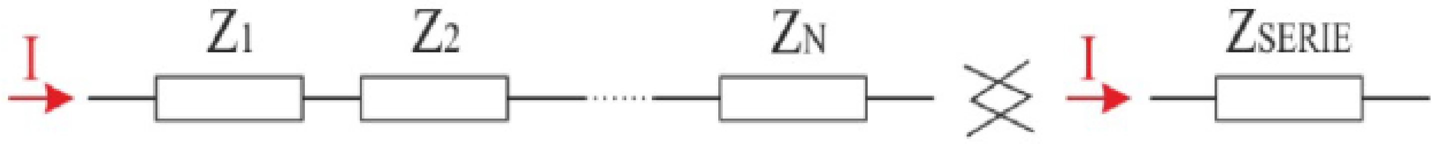
$$\left.\begin{array}{l}\mathbf{I}_1 = 2.84\mathrm{e}^{j0.12}\mathrm{mA} \\ \mathbf{I}_2 = 2.19\mathrm{e}^{-j0.58}\mathrm{mA}\end{array}\right\} \Rightarrow \begin{cases} i_1(t) = 2.84\sin(10^6 t + 0.12)\mathrm{mA} \\ i_2(t) = 2.19\sin(10^6 t - 0.58)\mathrm{mA}\end{cases}$$

The source, *e*(t), was a "**sin**" function, therefore the solutions are also "**sin**" functions



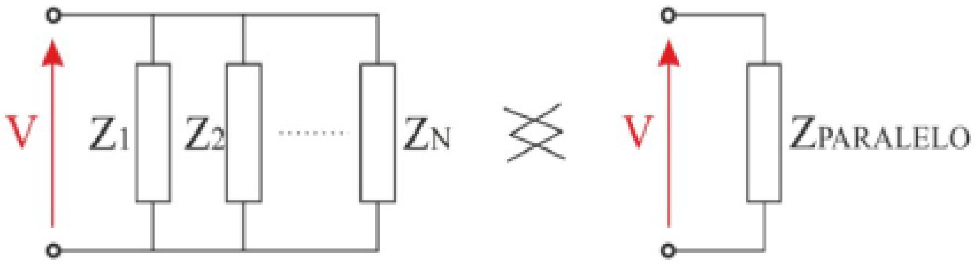
Association of Z's

- Series association



$$Z_{SERIE} = Z_1 + Z_2 + \cdots + Z_N = \sum_i Z_i$$

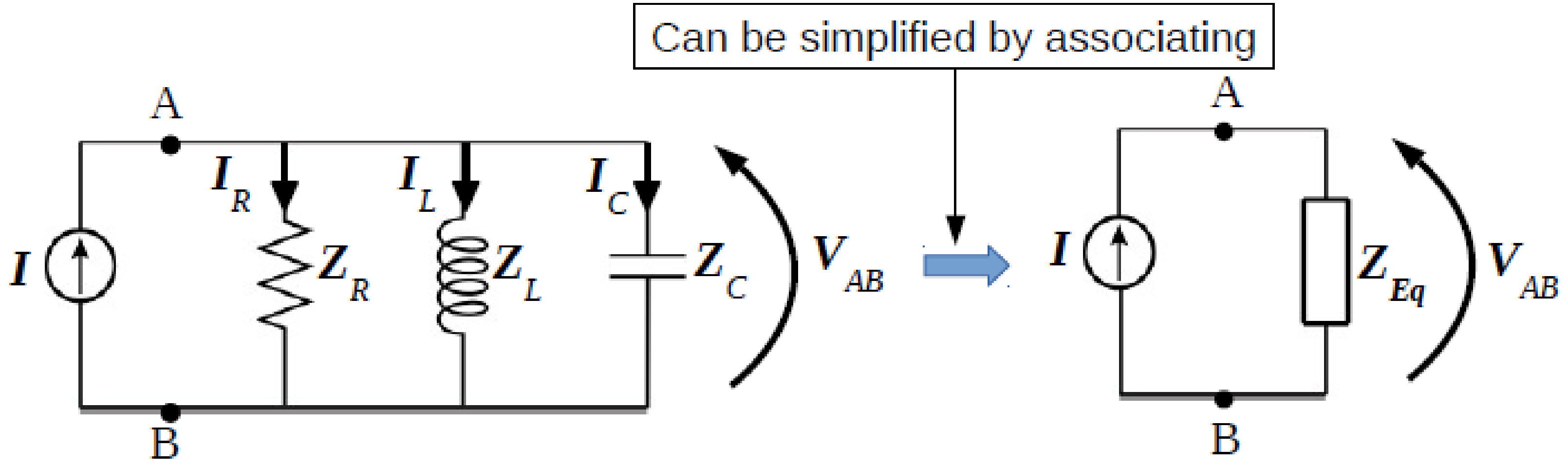
- Parallel association



$$\frac{1}{Z_{PARALELO}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N} = \sum_i \frac{1}{Z_i}$$

Example

- Calculate the currents trough each elements



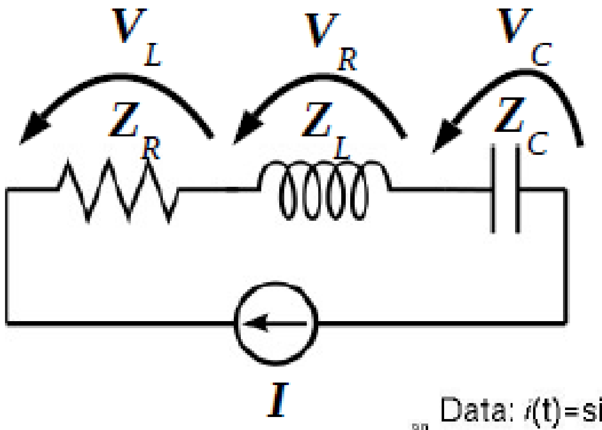
$$\mathbf{I}_R = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_R},\quad \mathbf{I}_L = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_L},\quad \mathbf{I}_C = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_C}$$

(The same voltage is applied to the three elements)

$$\begin{cases} \mathbf{V}_{AB} = \mathbf{Z}_{Eq}\mathbf{I} \\ \mathbf{Z}_{Eq} = \left(\frac{1}{\mathbf{Z}_R} + \frac{1}{\mathbf{Z}_L} + \frac{1}{\mathbf{Z}_C}\right)^{-1} \end{cases}$$

V and I relation in *R*, *L* and *C*

- Supposing a current $i(t) = I_0 \sin(\omega t + \phi_I)$ ➡ $\mathbf{I} = I_0 e^{j\phi_I}$



$$\mathbf{V}_R = \mathbf{Z}_R \mathbf{I} = R I_0 e^{j\phi_I}$$

V$_R$ and **I**$_R$ have the **same phase**

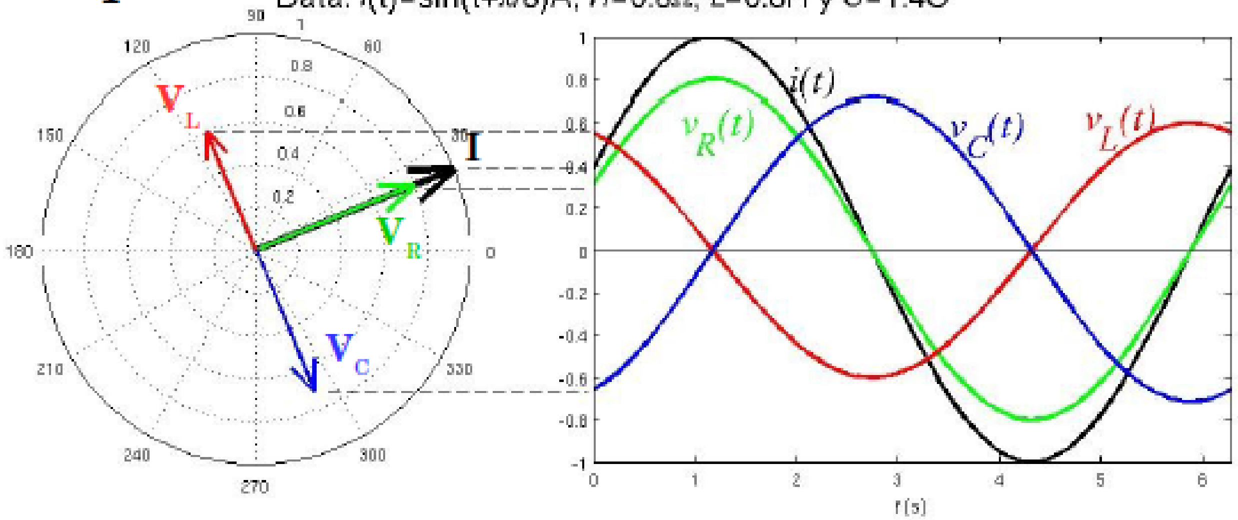
$$\mathbf{V}_L = \mathbf{Z}_L \mathbf{I} = j\omega L I_0 e^{j\phi_I} = \omega L I_0 e^{j(\phi_I + \frac{\pi}{2})}$$

V$_L$ is π/2 **ahead** to **I**$_L$

$$\mathbf{V}_C = \mathbf{Z}_C \mathbf{I} = \frac{1}{j\omega C} I_0 e^{j\phi_I} = \frac{I_0}{\omega C} e^{j(\phi_I - \frac{\pi}{2})}$$

V$_C$ is π/2 **behind** to **I**$_C$

Data: i(t)=sin(t+π/8)A, R=0.8Ω, L=0.6H y C=1.4C



Note that

$$\omega L I_0 = |\mathbf{V}_L|$$
$$\frac{I_0}{\omega C} = |\mathbf{V}_C|$$

are the amplitudes

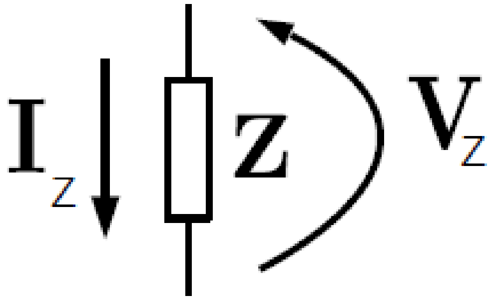
V and I relation for a generic Z

- For any other impedance, the phase between $\mathbf{V}_Z$ and $\mathbf{I}_Z$ is between -π/2 and π/2

$$\mathbf{Z} = \frac{\mathbf{V}_Z}{\mathbf{I}_Z} = \left|\frac{\mathbf{V}_Z}{\mathbf{I}_Z}\right| e^{j(\phi_V - \phi_I)} \Rightarrow \mathrm{Re}[\mathbf{Z}] = |\mathbf{Z}| \cos(\phi_V - \phi_I)$$

$\mathrm{Re}[\mathbf{Z}] \geqslant 0$ The real part of $\mathbf{Z}$ is the resistive part which is always ≥ 0

$$\Rightarrow -\frac{\pi}{2} \leqslant \phi_V - \phi_I \leqslant \frac{\pi}{2}$$



Power in SSS regimen

- In SSS regimen the power* in any electrical component is oscillating since it is the product of two oscillating signals: $p(t)=v(t)i(t)$. (The oscillation is then 2ω.) As we will see next,
  - The "ideal" elements *C* and *L* are storing and releasing the energy but **not dissipating** it.
  - The resistor *R* is the passive element that **dissipates** the energy although in an oscillating way.
  - For a impedance $\mathbf{Z}$, it is the **real part** that produces the energy dissipation.
  - The sources can **deliver or absorb** more or less energy depending on the phase difference between $v(t)$ and $i(t)$.
- To evaluate the power with a constant value in AC, it has to be **averaged over one period**

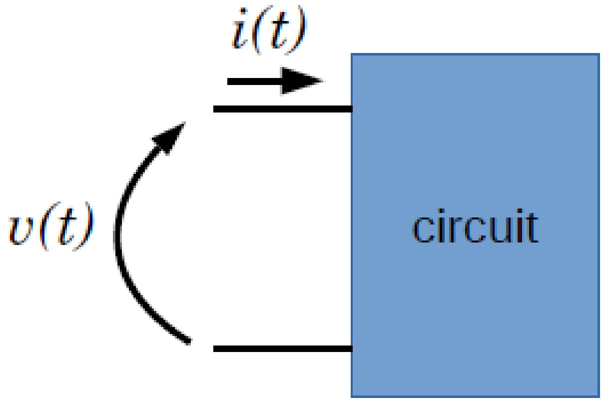
(*) power = energy per unit of time

Power in SSS regimen

- The *T*-averaged Power is then used in AC signals:

$$P = \frac{1}{T}\int_0^T v(t)i(t)dt$$

For the following two generic SSS signals with the indicated senses



$$\begin{cases} v(t) = V_0 \cos(\omega t + \phi_V) \\ i(t) = I_0 \cos(\omega t + \phi_I) \end{cases}$$

$$\Rightarrow P = \frac{1}{T}\int_0^T \frac{1}{2}V_0 I_0 \cos(\phi_V - \phi_I)dt + \cancelto{0}{\frac{1}{T}\int_0^T \frac{1}{2}V_0 I_0 \cos(2\omega t + \phi_V + \phi_I)dt}$$

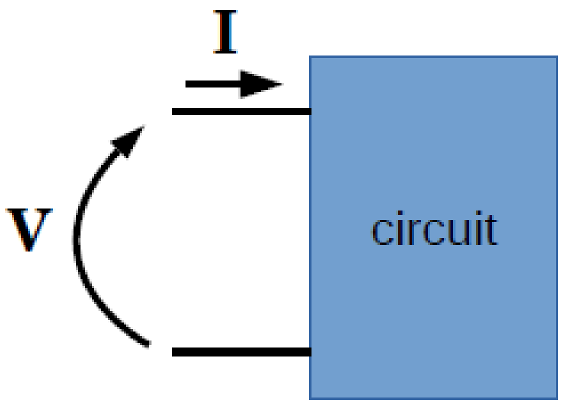
Power in SSS regimen

- For SSS signals the average power becomes

$$P = \frac{1}{2} V_0 I_0 \cos(\phi_V - \phi_I)$$

Power factor

- The power can be obtained from the phasors



$$\mathbf{V} = V_0 e^{j\phi_V} = a + jb$$

$$\mathbf{I} = I_0 e^{j\phi_I} = c + jd$$

(Scalar product)

$$\Rightarrow P = \frac{1}{2} \langle \mathbf{V}\mathbf{I} \rangle = \frac{1}{2}(ac + bd)$$

Power in *R*, *L* and *C*

- For a resistor the power factor is one since voltage in *R* and current trough *R* have the same phase*!!

$$P_R = \frac{1}{2} V_{0,R} I_{0,R} \cos(0) = \frac{1}{2} V_{0,R} I_{0,R} = \frac{1}{2} R I_{0,R}^2 = \frac{1}{2} R |\mathbf{I}_R|^2$$

- For *L* and *C* the power factor is zero since the current trough *L* or *C* and the voltage at *L* or *C* have a phase difference of π/2*

$$P_C = P_L = \frac{1}{2} V_0 I_0 \cos(\pm\frac{\pi}{2}) = 0$$

(*) remember slide 19

Power dissipated by Z

- For a generic passive element, $\mathbf{Z}$, only the real part dissipates power.

  From slide 20 we had

$$\mathbf{Z} = \left|\frac{\mathbf{V}}{\mathbf{I}}\right| \mathrm{e}^{j(\phi_V - \phi_I)} = \frac{V_0}{I_0}\left(\cos(\phi_V - \phi_I) + j\sin(\phi_V - \phi_I)\right)$$

  the real part of $\mathbf{Z}$ appears in *P*:

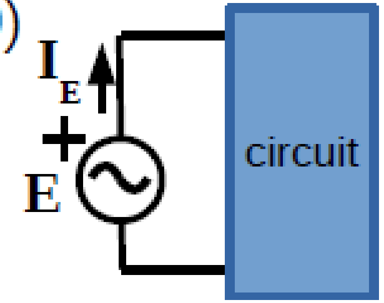
$$P_{\mathbf{Z}} = \frac{1}{2} V_0 I_0 \cos(\phi_V - \phi_I) = \frac{1}{2}\mathrm{Re}[\mathbf{Z}] I_0^2 = \frac{1}{2}\mathrm{Re}[\mathbf{Z}]|\mathbf{I}|^2$$

Power supplied/dissipated by a source

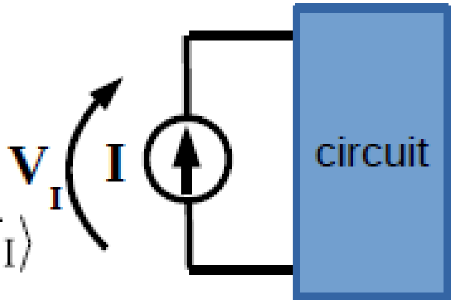
- There is a criteria for the sign of *P* depending on the relative sense of the current and voltage (as in DC)

$\mathbf{I}_\mathbf{E}$ and $\mathbf{E}$ in the same direction: The source is supposed to supply power (>0)

$$P_\mathrm{E} = \frac{1}{2}\langle \mathbf{E}, \mathbf{I}_\mathrm{E} \rangle$$

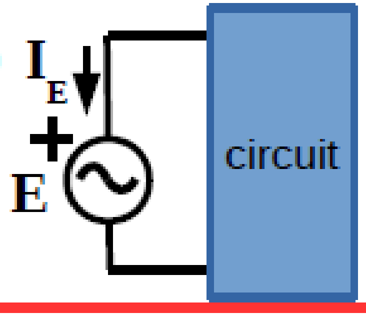


$$P_\mathrm{I} = \frac{1}{2}\langle \mathbf{I}, \mathbf{V}_\mathrm{I} \rangle$$

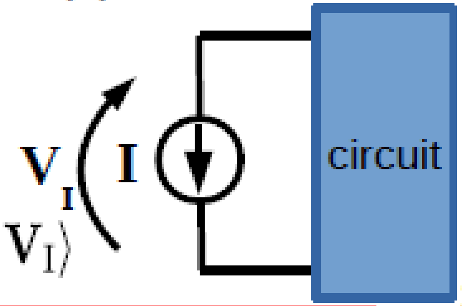


$\mathbf{I}_\mathbf{E}$ and $\mathbf{E}$ in opposite direction: The source is supposed to dissipate power (<0)

$$P_\mathrm{E} = -\frac{1}{2}\langle \mathbf{E}, \mathbf{I}_\mathrm{E} \rangle$$



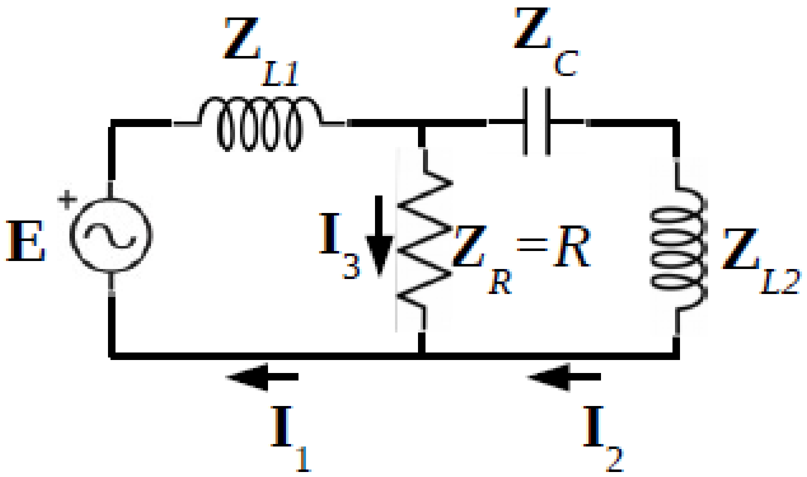
$$P_\mathrm{I} = -\frac{1}{2}\langle \mathbf{I}, \mathbf{V}_\mathrm{I} \rangle$$



The final sign of *P*, >0 or <0, tells if the source supplies or dissipates power

Example

- For the example in slide 15



$\mathbf{E} = j3\text{V} \qquad R = 300\Omega$

$\mathbf{I}_1 = 2.82 + j0.33\text{mA}$

$\mathbf{I}_2 = 1.83 - j1.19\text{mA}$

$$P_\text{E} = \frac{1}{2}\langle \mathbf{E}, \mathbf{I}_1 \rangle = \frac{1}{2}(0 \cdot 2.82 + 3 \cdot 10^3 \cdot 0.33) = 497.35\text{mW}$$

$$\mathbf{I}_3 = \mathbf{I}_1 - \mathbf{I}_2 = 0.99 + j1.52\text{mA}$$

$$P_R = \frac{1}{2}R|\mathbf{I}_3|^2 = 497.35\text{mW}$$

Power balance !!

Effective and mean values

- There are two magnitudes that are commonly used when measuring AC signals.
  - Mean value (measures the off-set)

$$A_m = \frac{1}{T}\int_0^T x(t)dt$$

  - Effective value (provided by the multimeter)

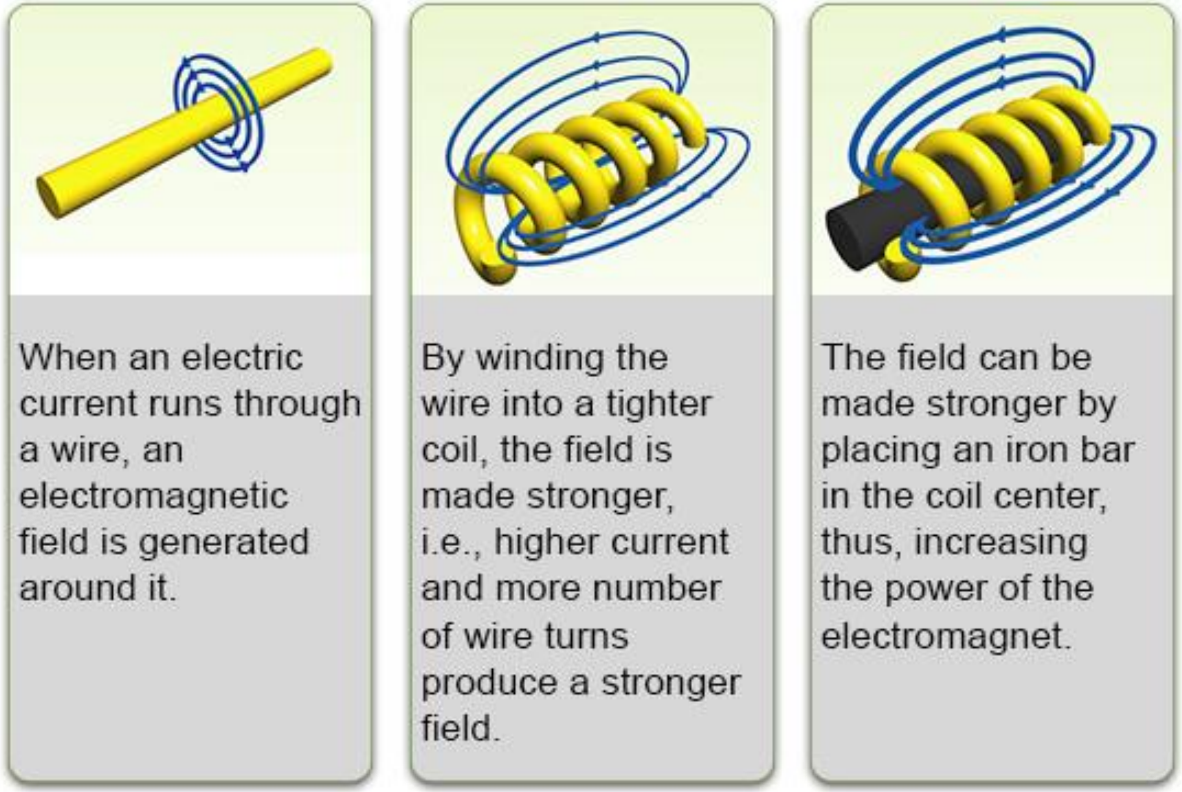
$$A_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T x(t)^2 dt}$$

For: $x(t) = A_0 \cos(\omega t) \quad \Rightarrow \quad \begin{cases} A_m = 0 \\ A_{eff} = \dfrac{A_0}{\sqrt{2}} \end{cases}$

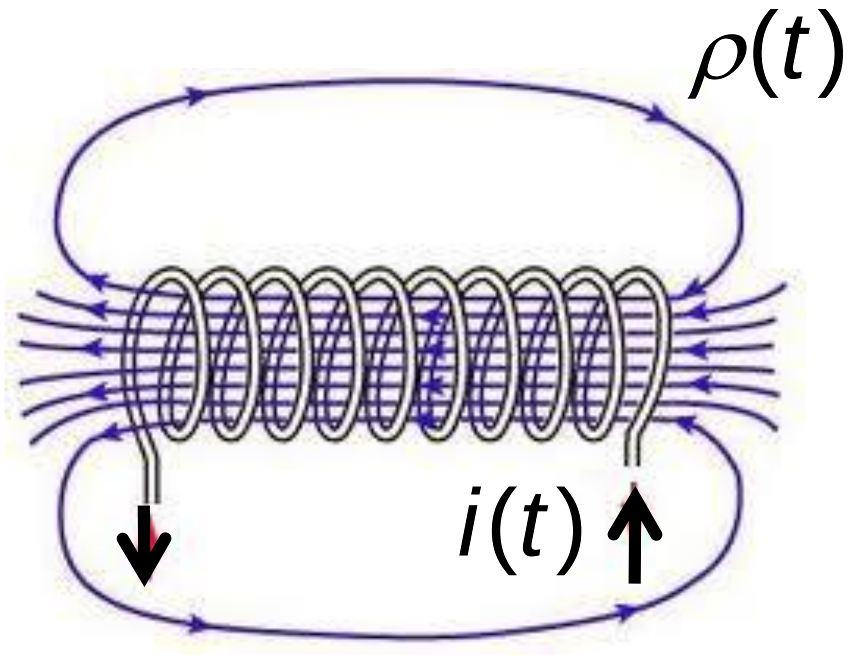
Magnetically coupled coils

Introduction

Ampere’s circuital law for a solenoid



Ampere's Law

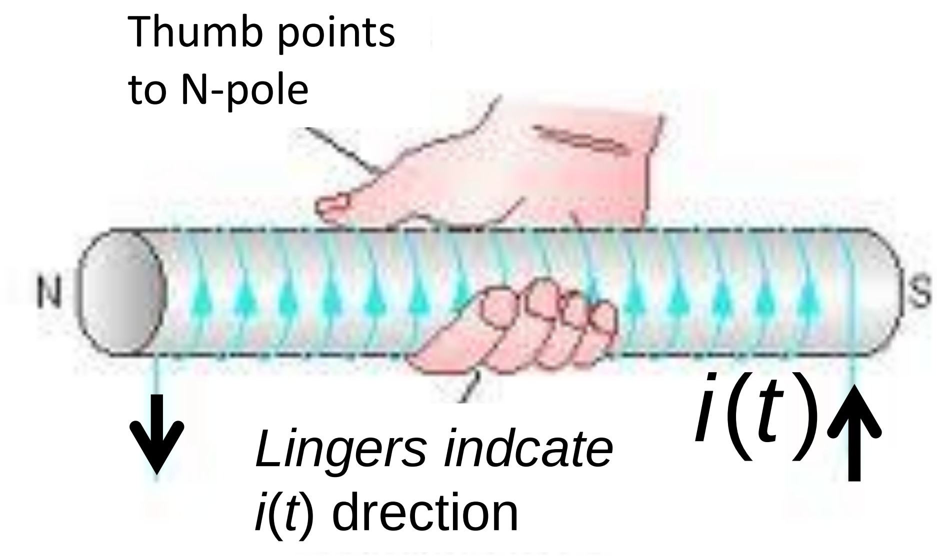


Right hand rule for the direction of $\rho(t)$:

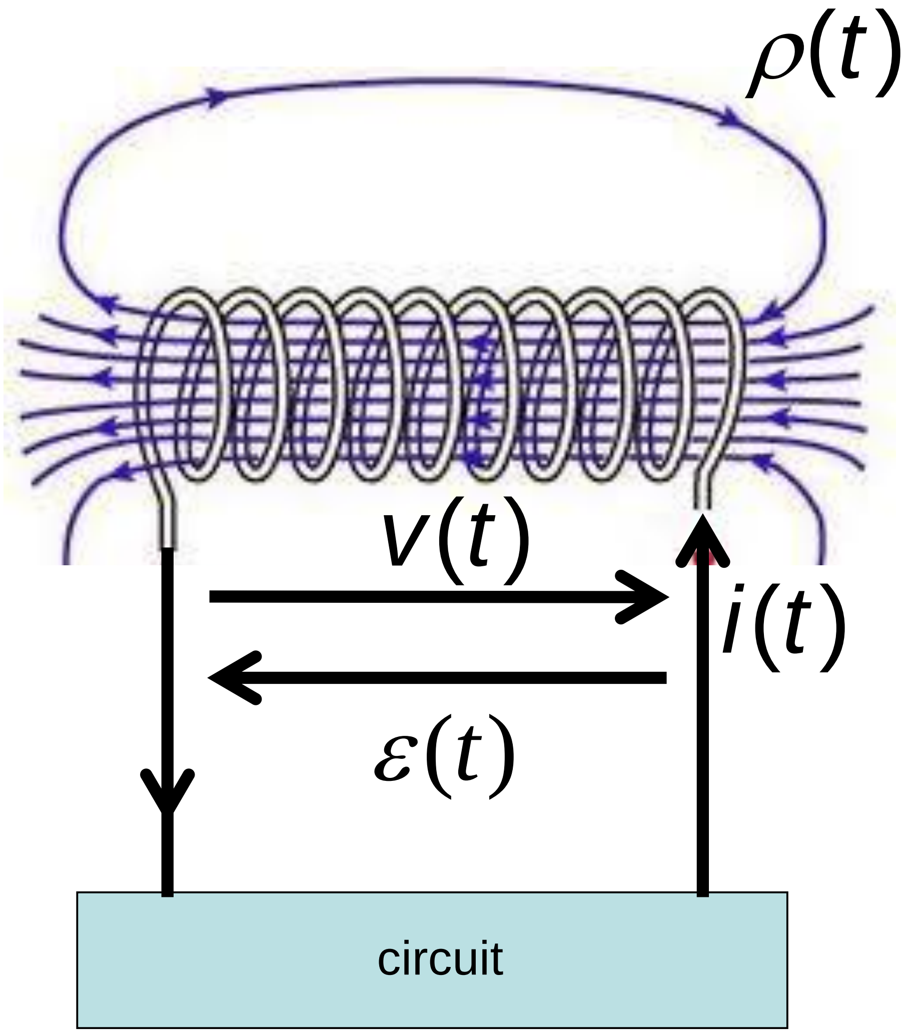
The current i(t) produces a magnetic flux $\rho(t)$:

$$\rho(t) = \frac{N}{\Re} i(t)$$

$\rho(t)$: magnetic flux (webers)
$i(t)$: electric current (amperes)
N: number of turns
$\Re$: reluctance (henries^{-1})



Faraday's law



The time varying flux $\rho(t)$ produces a voltage as shown in the figure

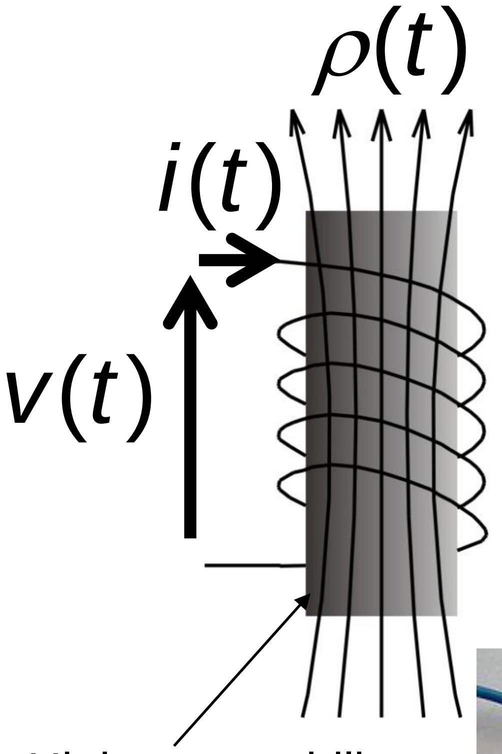
$$v(t) = N\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)$$

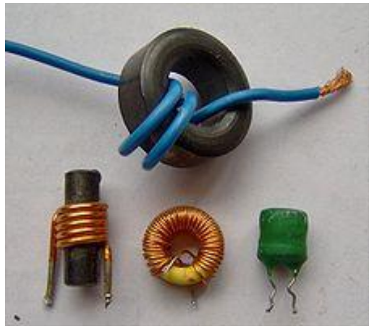
According to Len's law, the coil can be seen as an electromotive Force (applying a voltage to the circuit):

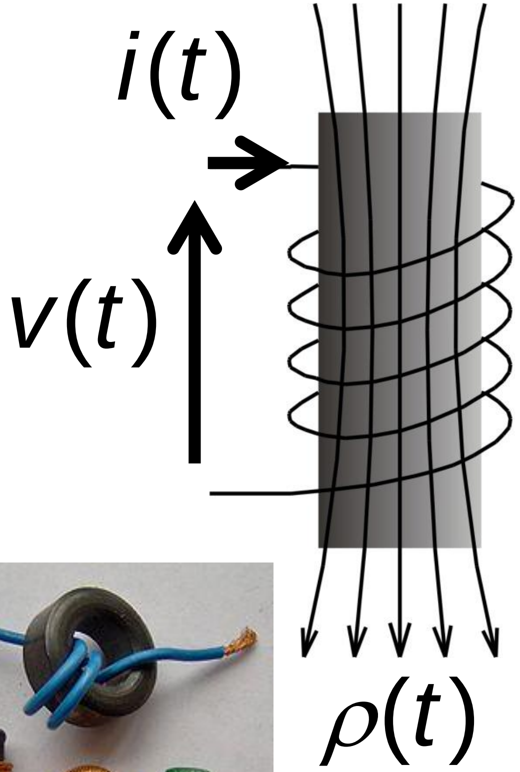
$$\varepsilon(t) = -N\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)$$

Ampere + Faraday

The time varying current *i*(t) produces a voltage *v*(*t*):







$$v(t) \overset{(*)}{=} L \frac{\mathrm{d}}{\mathrm{d}t} i(t)$$

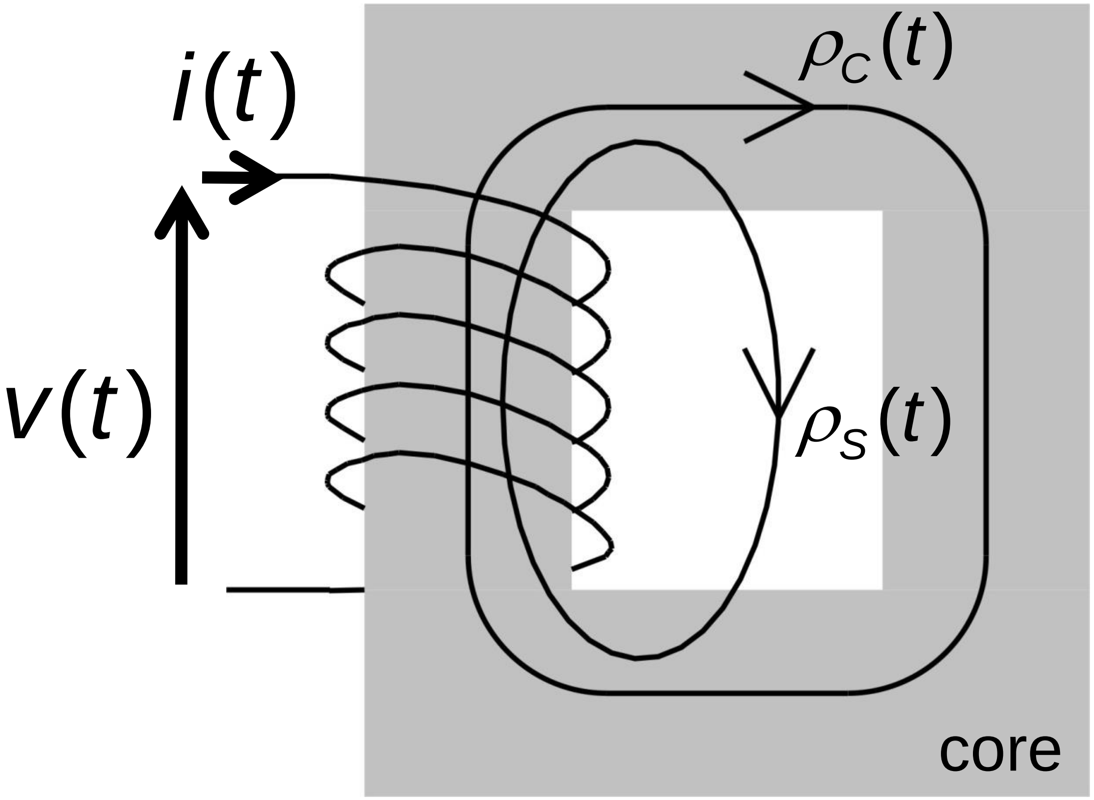
(*) Note that the voltage is pointing to the terminal whose current is entering the coil

being ***L*** the **self inductance** coefficient:

$$L := \frac{N^2}{\mathfrak{R}}$$

With units in henry (H)

Magnetic flux outside the coil



$$\rho(t) = \rho_C + \rho_S$$

$$= \left(\frac{1}{\Re_C} + \frac{1}{\Re_S} \right) Ni(t)$$

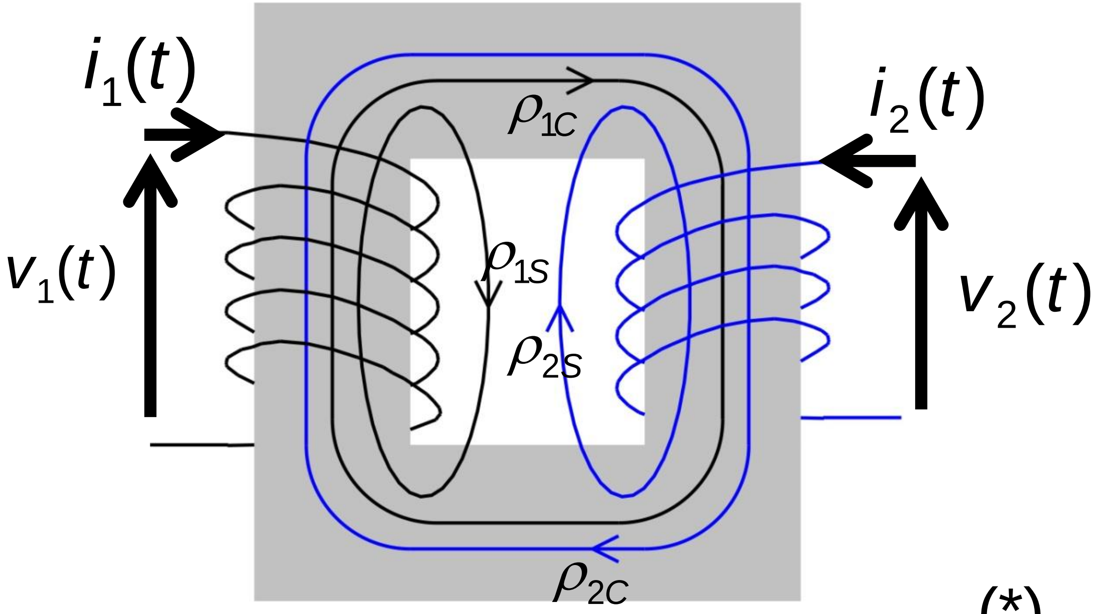
$\Re_C$: reluctance of the core,
$\Re_S$: leakage reluctance outside the core

For high-permeability core
$\Re_S >> \Re_C \Rightarrow \rho_C >> \rho_S$

Two magnetically coupled coils

- Voltage difference at the coils terminals.

(*)In this particular example the two magnetic flux inside the core generated by each coil have the same direction and are therefore added !!



(*)

$$\begin{cases} v_1 = N_1 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{1C} + \rho_{1S} + \rho_{2C}\right) \\ v_2 = N_2 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{2C} + \rho_{2S} + \rho_{1C}\right) \end{cases}$$

Two magnetically coupled coils

- Voltage difference at the coils terminals

$$\begin{cases} v_1 = N_1 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{N_1}{\Re_{1,C}} i_1 + \dfrac{N_1}{\Re_{1,S}} i_1 + \dfrac{N_2}{\Re_{2,C}} i_2\right) = N_1 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{N_1}{\Re_1} i_1 + \dfrac{N_2}{\Re_{2,C}} i_2\right) \\ v_2 = N_2 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{N_2}{\Re_{2,C}} i_2 + \dfrac{N_2}{\Re_{2,S}} i_2 + \dfrac{N_1}{\Re_{1,C}} i_1\right) = N_2 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{N_2}{\Re_2} i_2 + \dfrac{N_1}{\Re_{1,C}} i_1\right) \end{cases}$$

considering:

$$\frac{1}{\Re_1} = \frac{1}{\Re_{1,C}} + \frac{1}{\Re_{1,S}}, \quad \frac{1}{\Re_2} = \frac{1}{\Re_{2,C}} + \frac{1}{\Re_{2,S}}, \qquad \left(\begin{matrix}\text{We will supose that} \\ \Re_{1,C} = \Re_{2,C} = \Re_C\end{matrix}\right)$$

Two magnetically coupled coils

- Voltage difference at the coils terminals

$$\begin{cases} v_1 = L_1 \dfrac{\mathrm{d}}{\mathrm{d}t} i_1 + M_{12} \dfrac{\mathrm{d}}{\mathrm{d}t} i_2 \\ v_2 = L_2 \dfrac{\mathrm{d}}{\mathrm{d}t} i_2 + M_{21} \dfrac{\mathrm{d}}{\mathrm{d}t} i_1 \end{cases}$$

Self induced voltage

Mutual induced voltage

being:

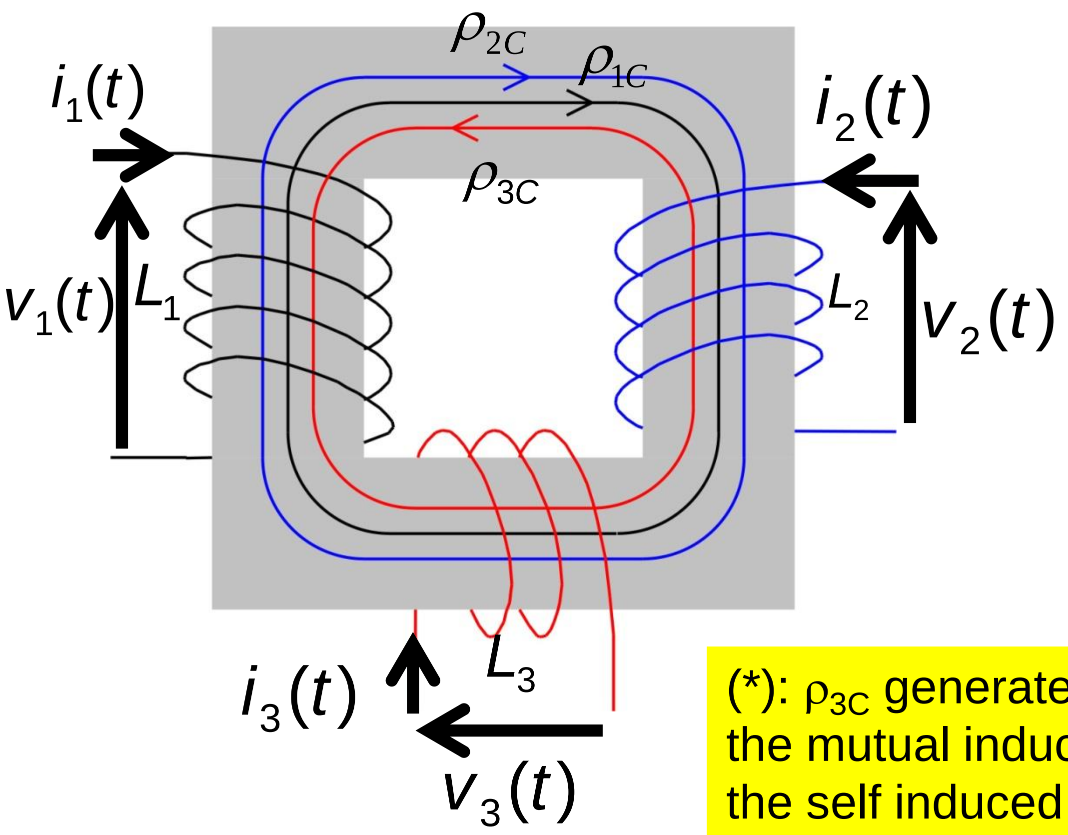
$L_1 = \dfrac{N_1^2}{\Re_1}$, the self inductance and

$M_{12} = \dfrac{N_1 N_2}{\Re_{2,C}}$, the **mutual inductance** of coil 2 into 1 (H)

$M_{12} = K_{12}\sqrt{L_1 L_2}$, where K_{12} is the **coupling coefficient** $(0 \le K_{12} \le 1)$

Three magnetically coupled coils

- Voltage diff. at the coils terminals in function of the currents i_1, i_2 and i_3.



$$\begin{cases} v_1 = L_1 \frac{\mathrm{d}}{\mathrm{d}t} i_1 + M_{12} \frac{\mathrm{d}}{\mathrm{d}t} i_2 - M_{13} \frac{\mathrm{d}}{\mathrm{d}t} i_3 \\ v_2 = L_2 \frac{\mathrm{d}}{\mathrm{d}t} i_2 + M_{21} \frac{\mathrm{d}}{\mathrm{d}t} i_1 - M_{23} \frac{\mathrm{d}}{\mathrm{d}t} i_3 \\ v_3 = L_3 \frac{\mathrm{d}}{\mathrm{d}t} i_3 - M_{31} \frac{\mathrm{d}}{\mathrm{d}t} i_1 - M_{32} \frac{\mathrm{d}}{\mathrm{d}t} i_2 \end{cases}$$

$$M_{12} = K_{12}\sqrt{L_1 L_2},$$

$$M_{23} = K_{23}\sqrt{L_2 L_3},$$

$$M_{13} = K_{13}\sqrt{L_1 L_3}.$$

(*): ρ_{3C} generated by coil 3 goes in opposite direction as ρ_{1C} therefore, the mutual induced voltage from coil 3 into coil 1 has opposite sign as the self induced voltage of coil 1

Three magnetically coupled coils

- For Sinusoidal Steady State regimen

$$v(t) = |V| e^{j\alpha} e^{j\omega t} = V e^{j\omega t}$$

$$i(t) = |I| e^{j\beta} e^{j\omega t} = I e^{j\omega t}$$

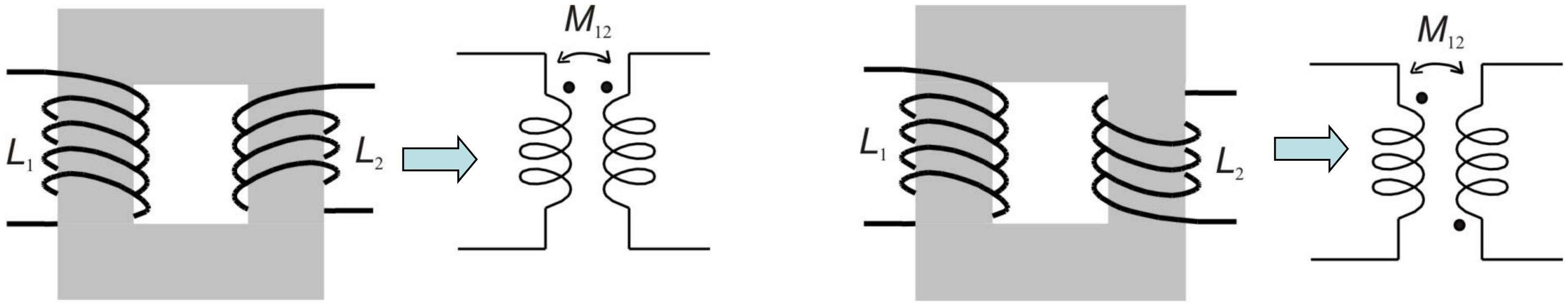
- Time differentiating

$$\begin{cases} V_1 = j\omega L_1 I_1 + j\omega M_{12} I_2 - j\omega M_{13} I_3 \\ V_2 = j\omega L_2 I_2 + j\omega M_{21} I_1 - j\omega M_{23} I_3 \\ V_3 = j\omega L_3 I_3 - j\omega M_{13} I_1 - j\omega M_{23} I_3 \end{cases}$$

$$\omega M_{12} = \omega K_{12} \sqrt{L_1 L_2} = K_{12} \sqrt{\omega L_1 \omega L_2}, (\text{units } \Omega)$$

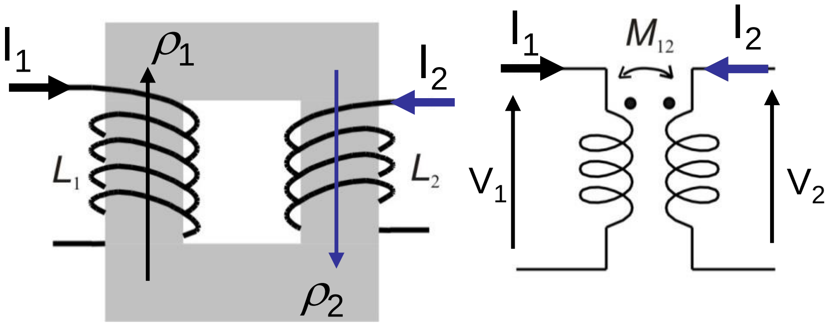
Dot’s convention

- Is used for simplifying the representation of coupled coils. It allows to know whether the mutually induced fluxes have the same direction or not, without need of drawing the way the spools are wounded around the core.
- Two coupled coils can be represented in one of the two following ways:

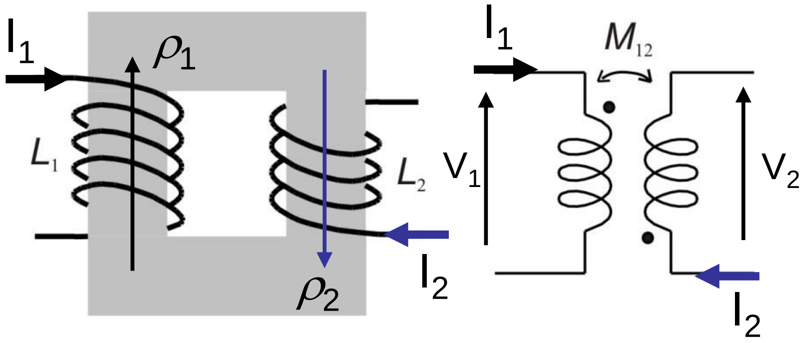


Dot’s convention

- If *both* currents entre or exit their respective coils by their dotted ends, then the generated fluxes inside the core have the same direction, otherwise the fluxes have opposite direction.



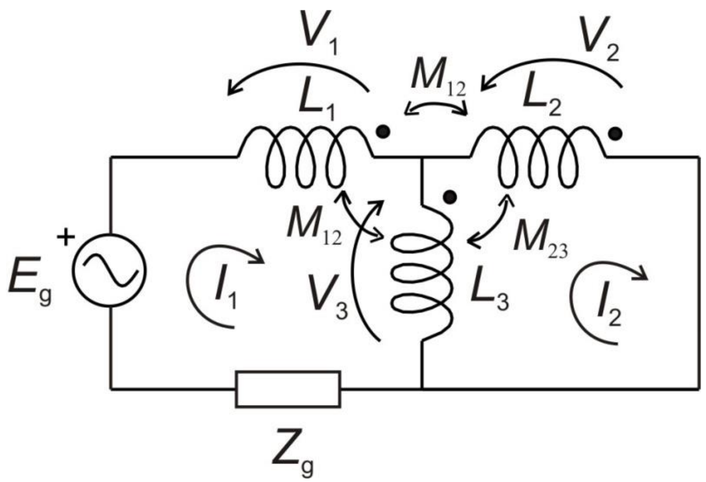
$$\begin{cases} V_1 = j\omega L_1 I_1 + j\omega M_{12} I_2 \\ V_2 = j\omega L_2 I_2 + j\omega M_{12} I_1 \end{cases}$$



$$\begin{cases} V_1 = j\omega L_1 I_1 + j\omega M_{12} I_2 \\ V_2 = -j\omega L_2 I_2 - j\omega M_{12} I_1 \end{cases}$$

Examples 1

- Equations for V_1,V_2 and V_3

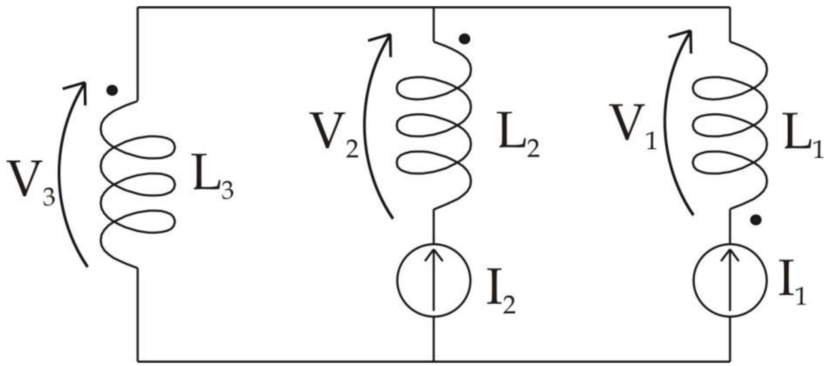


$$\begin{cases} E_g = V_1 + V_3 + Z_g I_1 \\ 0 = V_2 - V_3 \end{cases}$$

$$\begin{cases} V_1 = j\omega L_1 I_1 + j\omega M_{12} I_2 - j\omega M_{13}(I_1 - I_2) \\ V_2 = j\omega L_2 I_2 + j\omega M_{12} I_1 - j\omega M_{23}(I_1 - I_2) \\ V_3 = j\omega L_3 (I_1 - I_2) - j\omega M_{13} I_1 - j\omega M_{23} I_2 \end{cases}$$

Examples 2

- Equations for V_1,V_2 and V_3



$$\begin{cases} V_1 = -j\omega L_1 I_1 + j\omega M_{12} I_2 - j\omega M_{13}(I_1 + I_2) \\ V_2 = -j\omega L_2 I_2 + j\omega M_{12} I_1 + j\omega M_{23}(I_1 + I_2) \\ V_3 = j\omega L_3 (I_1 + I_2) + j\omega M_{13} I_1 - j\omega \omega_{23} I_2 \end{cases}$$