Module 4. Superposition theorem for circuits. Transformers

Circuit Theory, GIT
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Philip Siegmann
Departamento de Teoría de la Señal y Comunicaciones
Universidad de Alcalá

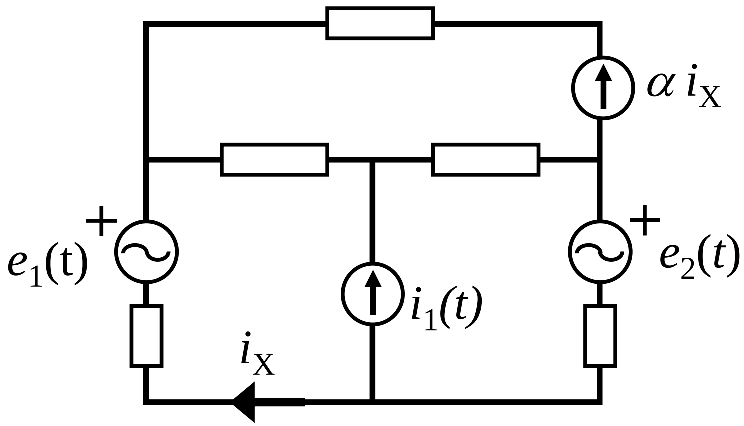
Fundamental theorems

- Superposition theorem
  - Principle of superposition
  - Multiplication by a constant
- Transformers
  - Real transformer
  - Perfect transformer
  - Ideal transformer

Principle of superposition

- Any linear circuit can be solved by considering each independent source acting by it's own:
  - Solve the circuit for each independent source
  - The solution of the global circuit is then the superposition of de solutions obtained by each source

Example 1:



$$e_1(t) = |E_1| \sin(\omega_1 t + \phi_1),$$
$$i_1(t) = |I_1| \sin(\omega_1 t + \theta_2),$$
$$e_2(t) = |E_2| \sin(\omega_2 t + \phi_2)$$

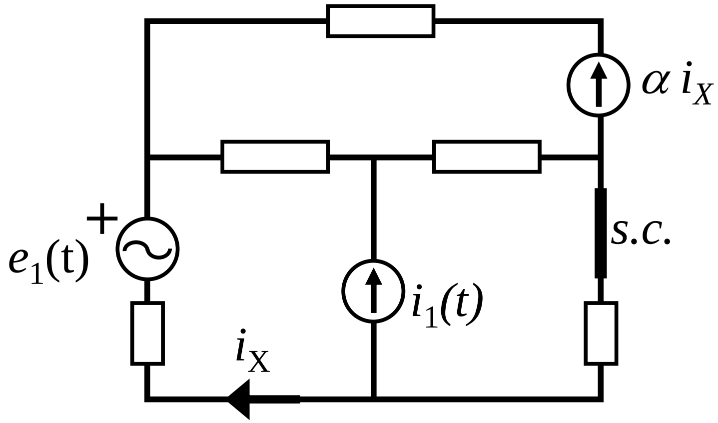
Principle of superposition

- One circuit for each source of the same temporal behavior (e.g. the same frequencies)
- **In**dependent sources of different temporal behavior are **annulled**:
  - Voltage sources replaced by s.c.
  - Current sources replaced by o.c.
- (Dependent sources are not annulled)

Principle of superposition

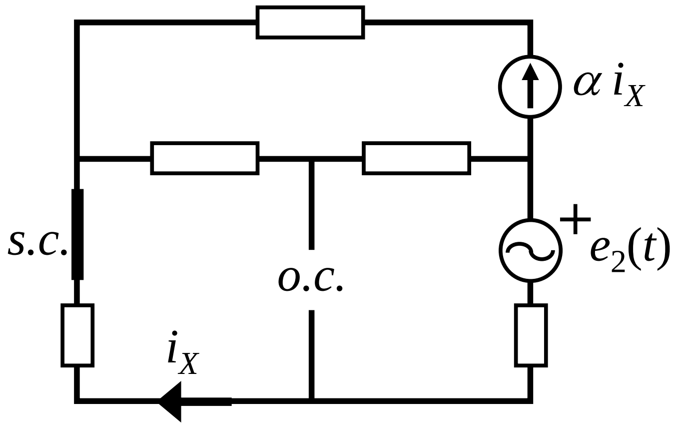
- Example 1: Two different kind of sources (ω_1, ω_2)

Circuit with sources of frequency $\omega = \boldsymbol{\omega_1}$



$$e_1(t) = |E_1| \sin(\boldsymbol{\omega_1} t + \phi_1),$$
$$i_1(t) = |I_1| \sin(\boldsymbol{\omega_1} t + \theta_2),$$
$$e_2(t) = 0$$

Circuit with sources of frequency $\omega = \boldsymbol{\omega_2}$



$$e_1(t) = 0,$$
$$i_1(t) = 0,$$
$$e_2(t) = |E_2| \sin(\boldsymbol{\omega_2} t + \phi_2)$$

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Principle of superposition

- Each circuit is solved independently for each temporal behavior (e.g. same frequency)
- For the **global circuit:**
  - The solution for the currents and voltages are added in the **time domain**:

$$i(t) = i(t)\big|_{\omega 1} + i(t)\big|_{\omega 2} + \ldots$$

$$v(t) = v(t)\big|_{\omega 1} + v(t)\big|_{\omega 2} + \ldots$$

This means obtained in the circuit with only sources of frequency ω_2

  - Te power delivered or dissipated by a **dependen source** is the addition of the power calculated in each circuit with a specific frequency:

$$P_{\text{dependen source}} = P_{\text{dependen source}}\Big|_{\omega 1} + P_{\text{dependen source}}\Big|_{\omega 2} + \ldots$$

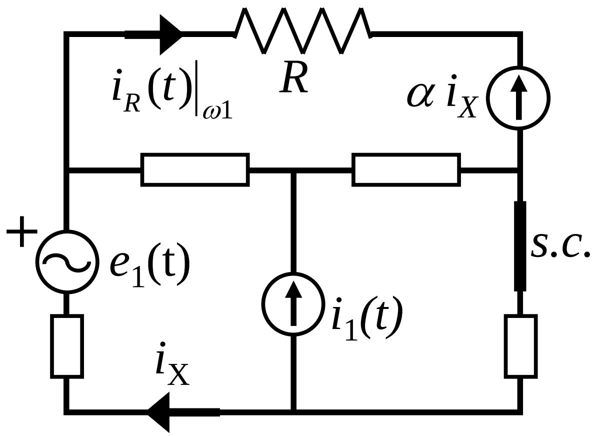
  - The power of an **independent source** of frequency ω_1 is calculated only for the circuit with sources of the same frequency:

$$P_{\text{independen source of frequency } \omega 1} = P_{\text{independen source}}\Big|_{\omega 1}$$

Principle of superposition

- Example 1: Two different kind of sources (ω_1, ω_2)

Circuit with sources of frequency $\omega = \boldsymbol{\omega_1}$



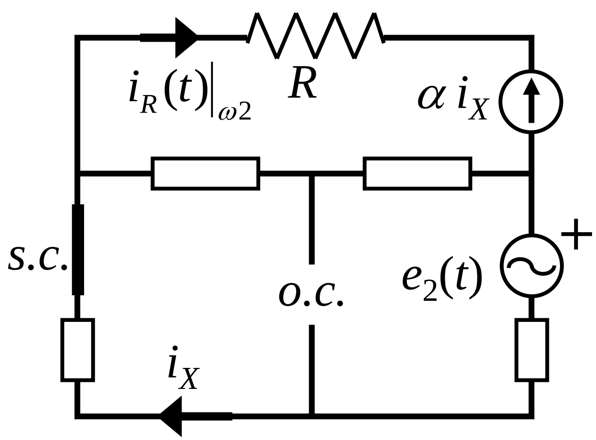
$$i_R(t)\big|_{\omega 1} = |I_1| \sin(\omega_1 t + \theta_1)$$

$$P_R\big|_{\omega 1} = \frac{1}{2} |I_R|^2_{\omega 1} R$$

$$P_{\alpha i_x}\big|_{\omega_1}$$

$$P_{e1} = P_{e1}\big|_{\omega 1}$$

Circuit with sources of frequency $\omega = \boldsymbol{\omega_2}$



$$i_R(t)\big|_{\omega 2} = |I_2| \sin(\omega_2 t + \theta_2)$$

$$P_R\big|_{\omega_2} = \frac{1}{2} |I_R|^2_{\omega 2} R$$

$$P_{\alpha i_x}\big|_{\omega_2}$$

$$P_{e2} = P_{e2}\big|_{\omega 2}$$

Superposition:

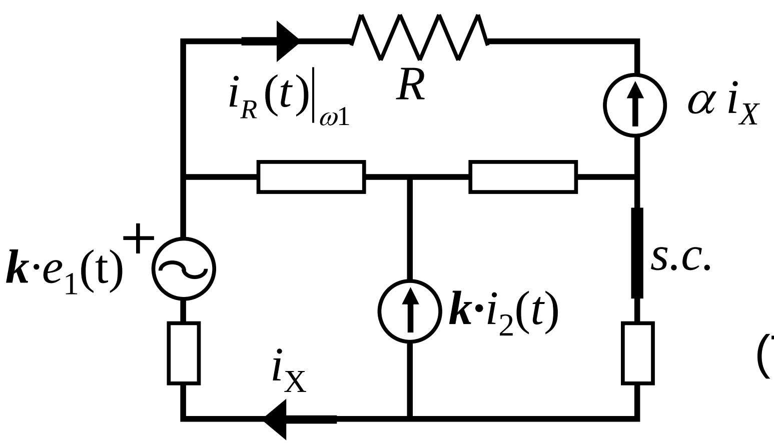
$$\begin{cases} i_R(t) = |I_1| \sin(\omega_1 t + \theta_1) + \\ \quad |I_2| \sin(\omega_2 t + \theta_2) \\ P_R = P_R\big|_{\omega_1} + P_R\big|_{\omega_2} \\ P_{\alpha i_x} = P_{\alpha i_x}\big|_{\omega_1} + P_{\alpha i_x}\big|_{\omega_2} \end{cases}$$

It can be used to separately analyze the DC and AC behavior of a circuit

Multiplication by a constant

- In a linear circuit, if **all the independent sources** with the same temporal behavior are multiplied by a constant, ***k***, then the current (or voltage) along any wire (or between any two points) in the circuit is multiplied by the same constant.

Example 1. Circuit with sources of frequency ω=ω1 are multiply by ***k***



You only need to multiply by ***k*** the previous result

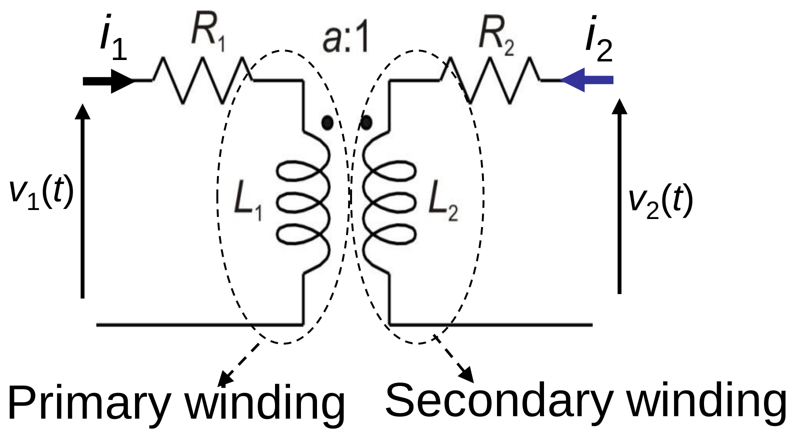
$$i_R(t) = \mathbf{k}\left|I_1\right|\sin(\omega_1 t + \theta_1) + \left|I_2\right|\sin(\omega_2 t + \theta_2)$$

(The previous calculated powers vary accordingly)

Transformers

Real transformer

- The transformers transfer electrical energy from one circuit to another one trough magnetically coupled coils



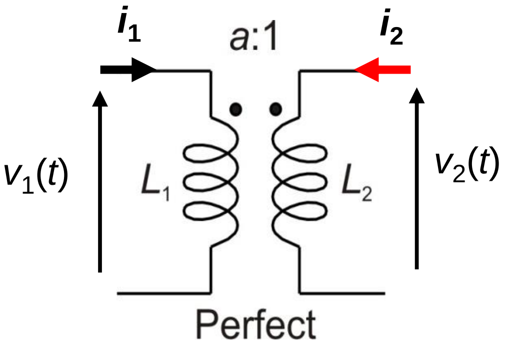
$$\begin{cases} R_1, R_2 \neq 0 \\ K \neq 1 \\ N_1, N_2 \text{ are finite} \end{cases}$$

$$a = \frac{N_1}{N_2} \text{ is the } \textbf{turn ratio}$$

$$\begin{cases} v_1(t) = \left(R_1 + L_1 \frac{\mathrm{d}}{\mathrm{d}t}\right) i_1(t) + M_{12} \frac{\mathrm{d}}{\mathrm{d}t} i_2(t) \\ v_2(t) = M_{12} \frac{\mathrm{d}}{\mathrm{d}t} i_1(t) + \left(R_2 + L_2 \frac{\mathrm{d}}{\mathrm{d}t}\right) i_2(t) \end{cases}$$

Perfect transformer

- The coupling coefficient is one and the number of turns are finite

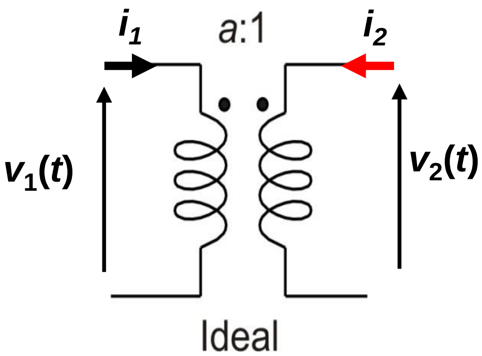


$$\begin{cases} R_1, R_2 = 0 \\ K = 1 \rightarrow \rho_S \cong 0 \\ N_1, N_2 \text{ are finite} \end{cases}$$

$$\begin{cases} v_1(t) = N_1 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{1,C} + \rho_{2,C}\right) \\ v_2(t) = N_2 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{2,C} + \rho_{1,C}\right) \end{cases} \Rightarrow \boxed{\frac{v_1(t)}{v_2(t)} = \frac{N_1}{N_2}} = \sqrt{\frac{L_1}{L_2}} = a.$$

Ideal transformer

- The coupling coefficient is one and the number of turns are infinite (but not the turn ratio)



$$\begin{cases} R_1, R_2 = 0 \\ K = 1 \\ N_1, N_2 \to \infty \end{cases}$$

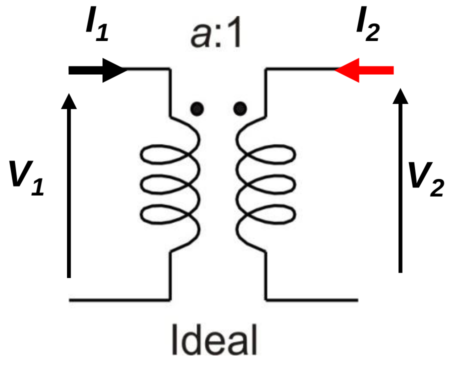
$$\begin{cases} v_1(t) = N_1 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{1,C} + \rho_{2,C}\right) \\ v_2(t) = N_2 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{2,C} + \rho_{1,C}\right) \end{cases}$$

- For Sinusoidal Steady State:

$$\begin{cases} V_1 e^{j\omega t} = N_1 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{1,C} e^{j\omega t} + \Phi_{2,C} e^{j\omega t}\right) = j\omega \underbrace{N_1}_{\to\infty} \overbrace{\left(\Phi_{1,C} + \Phi_{2,C}\right)}^{\to 0} e^{j\omega t} \\ V_2 e^{j\omega t} = N_2 \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{2,C} e^{j\omega t} + \Phi_{1,C} e^{j\omega t}\right) = j\omega N_2 \left(\Phi_{2,C} + \Phi_{1,C}\right) e^{j\omega t} \end{cases}$$

Ideal transformer

- The magnetic flux within the core becomes cero



$$\Phi_C = \Phi_{1C} + \Phi_{2C} \rightarrow 0$$

$$\Rightarrow \frac{N_1}{\Re_C} I_1 + \frac{N_2}{\Re_C} I_2 = 0$$

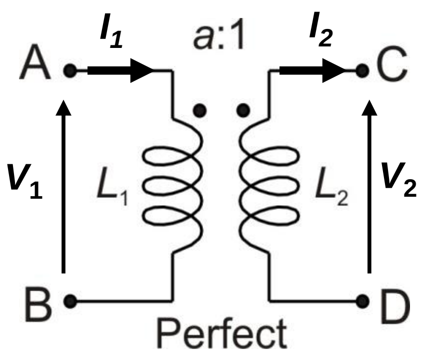
$$\Rightarrow \quad I_2 = -\frac{N_1}{N_2} I_1$$

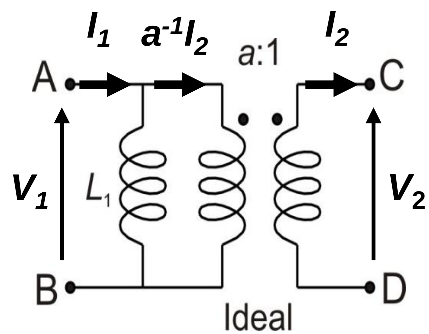
The mutual induced current in the secondary winding generates a magnetic flux that opposes and cancels the magnetic flux generated by the primary winding ⇒ The intensity exiting the dot at the secondary winding (I_2) is *a*-times the intensity entering trough the dot of the primary winding (I_1).

Perfect to Ideal transformer

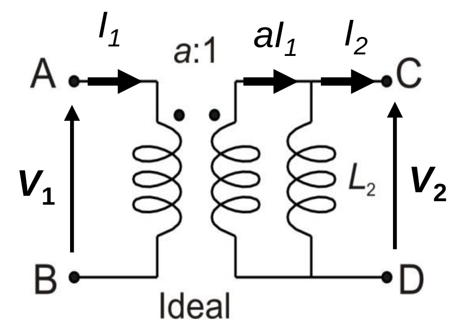
- Perfect transformer has an equivalent ideal transformer in parallel with an inductor

$$\begin{cases} V_1 = jwL_1\, I_1 - jwM_{12}\, I_2 \\ V_2 = -jwL_2\, I_2 + jwM_{12} \\ a = \sqrt{\dfrac{L_1}{L_2}},\ M_{12} = \sqrt{L_1 L_2} \Rightarrow M_{12} = L_1 \dfrac{1}{a} = L_2 a \end{cases}$$



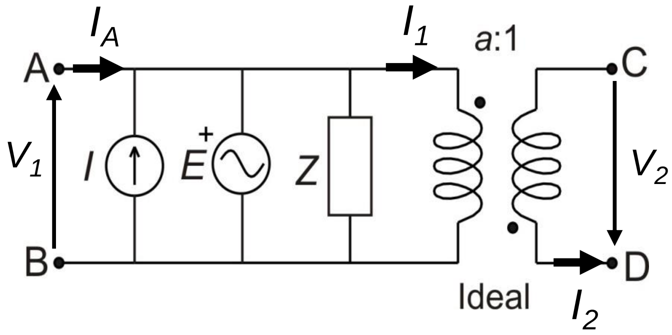


$$\Rightarrow \begin{cases} V_1 = j\omega L_1\left(I_1 - \dfrac{1}{a} I_2\right) \\ V_2 = j\omega L_2\left(aI_1 - I_2\right) \end{cases}$$



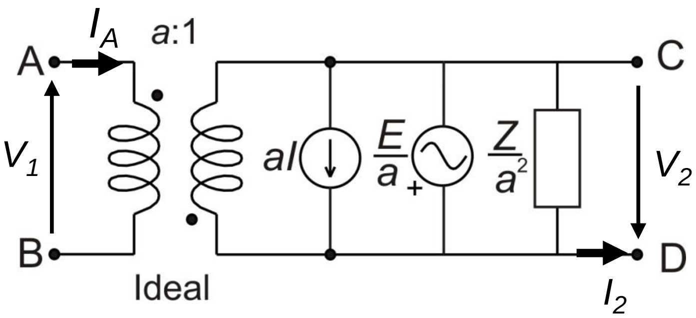
Operating with ideal transformers

- We can “move” elements form one side of the ideal transformer to the other side as follows



$$\begin{cases} I_2 = aI_1 \\ V_2 = \dfrac{1}{a}V_1 \end{cases}$$

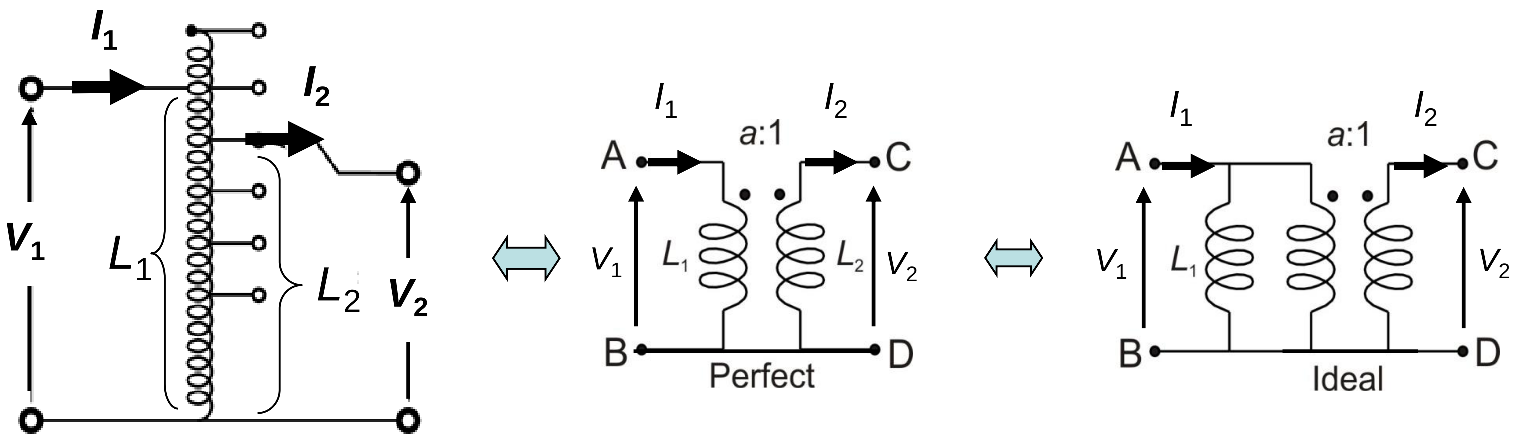
which is equivalent to:



Note: If the dot’s are crossed the direction of the electrical sources change

Autotransformer

- It is a perfect transformer with only one winding. Portions of the same winding act as both the primary and secondary winding (k=1)



$V_{AC} = V_1 - V_2 = V_1\left(1-\frac{1}{a}\right)$ is called the isolation voltage